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# NOTES ON MECHANICS.

DESIGNED TO BE USED IN CONNECTION WITH RANKINE'S APPLIED MECHANICS.

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## PART II.—DYNAMICS.

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# NOTES ON MECHANICS.

## DYNAMICS.

Dynamics is that part of Mechanics which considers forces as producing motion.

Velocity is the space (number of units of space) traversed by a moving body in one unit of time.

Motion is of two kinds, Uniform and Variable.

Motion is uniform when the velocity suffers no change, but remains constantly the same at all points of the body's path.

Motion is variable when the velocity is different at different points of the body's path. It is said to be Uniformly Varying when the velocity increases or decreases by equal amounts in equal times.

The amount of increase or decrease of a body's velocity in a unit of time is called its Acceleration.

## UNIFORM MOTION.

Let $t$ represent the time; $v$ the velocity; and $s$ the space through which a body moves in the time $t$; we then have the formula,

$$s = vt, \text{ or } v = \frac{s}{t}.$$

Thus if a body moves uniformly with a velocity of 12 feet per second for 5 seconds, the space traversed will be $s = 12 \times 5 = 60$ feet.

## VARIABLE MOTION.

When the velocity is variable we have no longer the formula, $v = \frac{s}{t}$, as the velocity changes during the time $t$, but if we take a very short time, so that the velocity may be considered nearly uniform during that interval, calling this short time $\Delta t$, and the space passed through in this time $\Delta s$; the quantity $\frac{\Delta s}{\Delta t}$ will nearly express the velocity of the body at that part of the path, and passing to the limit we shall have the true equation, $v = \frac{ds}{dt}$. The velocity at any point of the body's path means the velocity with which the body would move were all forces to stop acting when the body had reached the point in question.

## UNIFORMLY VARYING MOTION.

In this case the velocity receives equal increments in equal times, so that supposing a body to be already moving with a velocity $v_0$, and then a force which imparts to it an increase of velocity, or an acceleration represented by $f$ in each unit of time, it is evident that in $t$ units of time it will have a velocity,

$$v = v_0 + ft.$$

And since $v = \frac{ds}{dt}$, we have $\frac{ds}{dt} = v_0 + ft$; hence, by integration, we obtain $s = v_0 t + \frac{1}{2} ft^2$.

These are the equations of Uniformly Varying Motion. Differentiating the first of these equations, we have

$$\frac{dv}{dt} = f.$$

This is the amount of increase (or decrease) of velocity per unit of time, but $v = \frac{ds}{dt} \therefore \frac{dv}{dt} = \frac{d^2 s}{dt^2} = f$; or in words, the second differential coefficient of the space in regard to the time is equal to

the acceleration, or amount of change in the velocity per unit of time.

In the case of Variable Motion, $\frac{dv}{dt} = \frac{d^2s}{dt^2}$ still represents the change in velocity per unit of time, and hence the equation, $\frac{d^2s}{dt^2} = f$, is still true, with the difference that now $f$ is a variable, while in uniformly varying motion $f$ is a constant.

## MEASURE OF FORCES.

We have seen by the First Law of Motion, that a body at rest will remain at rest, and when in motion will continue to move uniformly and in a straight line, unless, and until, acted on by some external force.

We have also seen by the Second Law of Motion that every force which acts on a body produces its full effect, independently of any other force that may be acting on the same body. Considering these two facts together, we find, first, that the motion of a body is not changed as long as no external force acts upon it; and, secondly, that when a force does act it changes its motion.

Now suppose a body to be moving uniformly, and a force to act on the body for a certain length of time, $t$; in the direction of its motion the effect of the force is evidently to increase the velocity, and if $f$ represents the amount of velocity the force would impart in one unit of time, the total increase of velocity imparted at the end of time $t$, will be $ft$; and if the force stop acting the body will again move uniformly, but with a velocity greater by $ft$.

A force which will impart to the same body in the same time a velocity twice as great, is itself twice as great, and hence we can measure forces by the amounts of velocity they impart to the same body in the same time.

Next suppose two bodies, equal in every respect, moving side by side with the same velocity, and uniformly. Apply to one of them a force F, in the direction of the body's motion; the effect of this

force is to increase the velocity with which the body moves, and if we wish to give to the other an equal increase of velocity, so that they shall still move side by side, we must apply an equal force to that. Now let us unite these two bodies into one; it will still require a force F to give it the required increment of velocity. Hence we see that if we double the mass to which we wish to impart a certain velocity, we must double the force: or in other words, employ a force which would impart to the first mass a velocity double the required velocity.

Momentum is a term applied to the product obtained by multiplying the number of units of mass in a body by its velocity. It is merely the amount of velocity per unit of mass, and is that by which we measure forces, since we have already shown that they are proportional to the momenta (or the velocities per unit of mass), they generate in the same time in a body; hence if F represents the force, $m$ the mass of the body, and $v$ its velocity, we have,

$$F = mv.$$

Take now the particular case, when the direction of the force coincides with the direction of the body's motion, and we have,

$$F = m\frac{d^2s}{dt^2}.$$

## THIRD LAW OF MOTION.

The Third Law of Motion which was referred to in the Statics, is (as given by Sir Isaac Newton) that Action and Reaction are equal.

In other words, that momentum cannot be given to one body without giving an equal and opposite momentum to another.

Thus, if a hunter shoots a bird, he receives a harder blow than the bird, only it is distributed over a larger space, and hence the intensity of the blow is not so great.

One man cannot strike another without being struck by him, etc.

## UNIT OF MASS.

We have in Statics taken weights as the measures of forces, and we have now shown that forces are proportional to the mo-

menta they would generate in a unit of time, and now since we have not yet fixed on a unit of mass we may so choose it that these two measures (their numerical representatives) shall be equal.

Now it is found by experiment that the acceleration due to the force of gravity is constant at the same place on the earth's surface, and if this acceleration be represented by $g$, the momentum of a falling body of mass $m$ will be $mg$, which is the dynamical measure of the force of gravity acting on the body at that point.

W is its statical measure, and we so choose the unit of mass as to make these equal to each other, or

$$W = mg, \text{ hence } m = \frac{W}{g};$$

or, in other words, the mass of a body is equal to its weight divided by the acceleration due to gravity. If we make $m = 1$, we have $W = g$, or in words, The unit of mass is a mass whose weight is equal to the acceleration due to gravity, or generally, about $32\frac{1}{6}$ times the unit of weight.

## DECOMPOSITION OF FORCES.

We have seen that when a force acts on a body in the direction of its motion, the measure of the force is $F = m\frac{d^2s}{dt^2}$.

If, now, the line of motion of the body (Fig. 1) is $A\dot{C}$, and the force F acts on the body when it has arrived at the point B, the force being inclined to the direction of the body's motion at an angle $\theta$, the effect will be to turn the body out of its course, and if the force continue to act for any length of time the body will describe a curve. Now the force can be decomposed into two forces, one acting in the direction of the body's motion, and one perpendicular to its path; the component in the direction of the body's motion will be $F \cos\theta = m\frac{d^2s}{dt^2}$; the component perpendicular to its direction is $F \sin\theta$, and we must find a

measure for this force in terms of the velocity, and the radius of curvature of the path.

To do this we must observe, that whatever the path pursued by the body, by drawing the osculatory circle at the point of the path where the body is, we can imagine the body to be moving approximately on the arc of this circle. Let $A_1A_2$ (Fig 2) be the portion of this arc described in the time $\varDelta t$; and let $A_2v = v$; then (Rankine, Art. 363)

$$\text{Arc } A_1A_2 = v\varDelta t, \quad \text{chord } A_1A_2 = v\varDelta t \frac{\text{chord}}{\text{arc}};$$

$$\text{but } A_2V_2 : CA :: vV_2 : A_1A_2 \therefore$$

$$vV_2 = \frac{A_2V_2 \cdot A_1A_2}{CA} = \frac{v^2\varDelta t}{r} \frac{\text{chord}}{\text{arc}}$$

$$\therefore vV_2 = \frac{v^2}{r} \frac{\text{chord}}{\text{arc}} \varDelta t.$$

Now $vV_2$ represents a force which, compounded with $A_2v$, produces $A_2V_2$; hence it represents the amount of change in the velocity of the body in the time $\varDelta t$, approximately, $\therefore \frac{vV_2}{\varDelta t} = \frac{v^2}{r} \frac{\text{chord}}{\text{arc}}$ is approximately the amount of change in velocity per unit of time $\therefore$ the true amount of deviation is $\frac{v^2}{r}$, which we obtain by passing to the limit; hence if the mass is $m$, we have

$$F \sin \theta = m\frac{v^2}{r}.$$

Another and more general way to decompose forces is to assume, either two rectangular axes in the plane of the motion, if the path is of single curvature, or, in general, three rectangular axes, $Ox$, $Oy$, and $Oz$. We have already seen that $v = \frac{ds}{dt}$, and hence $vdt = ds$. Now this small space, $\varDelta s$, can be made the diagonal of a parallelopiped, whose sides will be $\varDelta x$, $\varDelta y$, $\varDelta z$; hence the components of the velocity will be $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$, respectively, then the components of the acceleration, or the accelerations in

the direction O$x$, O$y$ and O$z$, will be $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$ and $\frac{d^2z}{dt^2}$, for these are the limits of the ratios of the increments of velocity to the increments of time; hence if $\alpha$, $\beta$, and $\gamma$ denote the angles the force makes with O$x$, O$y$ and O$z$, we have,

$$\mathrm{F}\cos\alpha = m\frac{d^2x}{dt^2},\ \mathrm{F}\cos\beta = m\frac{d^2y}{dt^2}\text{ and }\mathrm{F}\ \cos\ \gamma = \frac{d^2z}{dt^2}.$$

## CONSIDERATION OF UNIFORM AND UNIFORMLY VARYING MOTION.

We have seen, from considering the laws of motion, that uniform motion is a state, or condition of equilibrium; that is, a body which is moving uniformly is either acted on by no force, or by balanced forces. The formulæ we have obtained for uniform motion, viz., $s = vt$, etc., are so simple as to need no farther discussion here.

We will now consider uniformly varying motion, and supposing the body to be moving in a straight line under the action of a constant force, F, we shall have, $\mathrm{F} = m\frac{d^2s}{dt^2}$ $\therefore$ $\frac{\mathrm{F}}{m} = \frac{d^2s}{dt^2}$ or if

$$\frac{\mathrm{F}}{m} = f,\ f = \frac{d^2s}{dt^2}\ \therefore\ \frac{ds}{dt} = ft + v_0,\text{ and } s = v_0t + \tfrac{1}{2}ft^2,$$

as previously developed, where $f$ denotes the actual acceleration of the body per unit of time. As a useful case, we will apply this to falling bodies, and those thrown vertically upward. In either case, the only force acting is that of gravity, and the acceleration due to gravity we represent by $g$, which is nearly equal to $32\frac{1}{6}$. We shall then have the general formula,

$$h = v_0t + \tfrac{1}{2}gt^2,\text{ and } v = gt.$$

That is, the vertical height traversed by a falling body in any time $t$, is $v_0t + \frac{1}{2}gt^2$, where $v_0$ represents the velocity it had at the beginning of the time. If the time be reckoned from when the body is at its highest point, we have $v_0 = 0$, and then $h = \frac{1}{2}gt^2$, and $v = gt$.

EXAMPLES.

1. A stone is dropped down a precipice, and is heard to strike the bottom in 4 seconds after it started, how deep is the precipice?

*Solution.* Here $h = \frac{1}{2}g(4)^2 = 8(32\frac{1}{6}) = 257\frac{1}{3}$ feet.

2. How long will a stone, dropped from the top of a precipice, 500 feet deep, take to reach the bottom?

3. What velocity will the stone, in the first example, have acquired when it reaches the bottom of the precipice?

4. What will be that of the stone in the second example?

5. A body is thrown vertically upward with a velocity of 100 feet per second, to what height will it rise?

*Solution.* $h = v_0t - \frac{1}{2}gt^2$ (since gravity acts in a direction opposite to the body's motion) $\therefore h = 100t - \frac{1}{2}(32\frac{1}{6})t^2$, also when it has reached the highest point it must have lost the velocity $v = 100 \therefore 100 = gt \therefore t = \frac{100}{32\frac{1}{6}} = 3.1088$ sec., and $h = 100\,t - \frac{1}{2}(32\frac{1}{6})t^2 = 155.44$ feet. The body will now begin to descend, and, to reach the point from which it started, will take a time equal to 3.1088 sec., and when it reaches that point will have acquired a downward velocity, equal to 100 feet per second again.

6. A body is thrown vertically upwards, and rises to a height of 50 ft., with what velocity was it thrown, and how long was it in its ascent?

7. What velocity will the same body acquire when it reaches a point 20 feet below the point from which it was thrown?

8. What will be its velocity in its ascent at a point 15 feet above the point where it was thrown, and what that at the same point in its descent?

The two equations, $h = \frac{1}{2}gt^2$ and $v = gt$, give, by eliminating $t$, $h = \frac{1}{2}\frac{v^2}{g}$, or $v^2 = 2gh \therefore v = \sqrt{2gh}$; that is, in words, A body will, in falling through a height $h$, acquire a velocity, $v = \sqrt{2gh}$; thus if a body falls through a height of 50 feet, it will, by that fall, acquire a velocity of $\sqrt{2(32\frac{1}{6})50} = \sqrt{3216.26} = 56.7$ feet per second.

This is called the velocity due to the height $h$; so if a body has a velocity of 40 feet, we have $h = \frac{v^2}{2g} = \frac{1600}{64.3} = 24.8$, or we say the body has a velocity due to the height 24.8 feet; that is, a velocity which it would acquire by falling through a height of 24.8 feet.

## CONSTRAINED MOTION.

A body may be acted on by a force which gives it a motion in some definite direction, and be prevented from pursuing its course by the intervention of some other body; this is the case when a body is caused to move in a groove, or on an inclined plane.

In these cases, if the force acting on the body at any point of its path be resolved into two, one perpendicular to the surface on which the body is moving, and the other along the surface, the first of these components will be entirely destroyed by the resistance of the surface on which the body is moving. Thus suppose a body whose weight is W, to move down an inclined plane (Fig. 4) acted on by the force of gravity alone.

Let $\theta$ be the inclination of the plane to the horizon (Fig. 4). The force of gravity W, acting on the body, acts in a vertical line HF, and decomposing it into two components, HD and HE, respectively parallel and perpendicular to the path, we shall have the two components, $HD = W \sin\theta$, and $HE = W \cos\theta$, the last of which is entirely counteracted by the resistance of the plane (a force equal and opposite to HE), and the first force, $HD = W \sin\theta$, is the force that causes the motion. Since $\theta$ and W are both constant quantities, the force is constant, and hence produces uniformly accelerated motion, the acceleration being $g \sin\theta$; hence we have $v = g \sin\theta{\cdot}t$, and $s = \frac{1}{2}g \sin\theta{\cdot}t^2$; or if the body had an initial velocity, $v_0$, we shall have the formulæ, $v = \pm v_0 + g \sin\theta{\cdot}t$, $s = \pm v_0t + \frac{1}{2}g \sin\theta{\cdot}t^2$, according as the initial velocity is down or up the plane.

I. Suppose the body to have no initial velocity, but to fall from A

to B, by the force of gravity alone, we then have, as above, the formulæ,

$$v = g \sin\theta \cdot t, \text{ and } s = \tfrac{1}{2} g \sin\theta \cdot t^2.$$

$$\therefore t^2 = \frac{2s}{g \sin\theta} \therefore t = \sqrt{\frac{2s}{g \sin\theta}} = \sqrt{\frac{2\text{AB}}{g \sin\theta}}.$$

Now suppose another body to fall freely from A to C, we should have, $\text{AC} = \frac{1}{2} g t'^2 \therefore t' = \sqrt{\frac{2\text{AC}}{g}} \therefore t : t' = \sqrt{\frac{2\text{AB}}{g \sin\theta}} : \sqrt{\frac{2\text{AC}}{g}}$ $= \sqrt{\frac{\text{AB}}{\sin\theta}} : \sqrt{\text{AC}}$; $\therefore t : t' \sqrt{\frac{\text{AB}^2}{\text{AC}}} : \sqrt{\text{AC}} = \text{AB} : \text{AC}$, since $\sin\theta = \frac{\text{AC}}{\text{AB}}$. Hence the time down the plane, is to the time down the vertical, as the length is to the height.

II. From the same formulæ, if $v$ denote the velocity at the foot of the plane, *i. e.*, at B, and $v'$ that at C, we have $v = g \sin\theta \cdot t = g \sin\theta \sqrt{\frac{2\text{AB}}{g \sin\theta}} = \sqrt{2\text{AB}\, g \sin\theta} = \sqrt{2\text{AC} \cdot g}$, and $v' = g t' = g\sqrt{\left(\frac{2\text{AC}}{g}\right)} = \sqrt{2\text{AC} \cdot g}$.

Hence the velocity at the foot of the plane is equal to that at the foot of the vertical.

This shows that the velocitiy acquired by a falling body, in falling through a certain height $h$, is precisely the same whether the path is vertical, or inclined, viz. $\sqrt{2gh}$.

Hence if the body be obliged to pursue a broken line. or a curved line, in its descent, when it has fallen through a height $h$, the velocity it has acquired will still be $\sqrt{2gh}$.

To find the time down a curved line. Suppose a body acted on by gravity to be constrained to move on the curve AC (Fig. 5), in falling from A to P through a height BD, the velocity acquired will be $\sqrt{2g\text{BD}} = \sqrt{2gx}$, and the space PP′ described in the next short interval of time $\Delta t$, will be $v \Delta t$, nearly,

$$\therefore \text{PP}' = \sqrt{2gx} \cdot \Delta t \therefore \Delta t = \frac{\text{PP}'}{\sqrt{2gx}} = \frac{\Delta s}{\sqrt{2gx}}, \text{ nearly, or}$$

$t = \int \frac{ds}{\sqrt{2gx}}$. This might be at once derived from the formula,

$\frac{ds}{dt} = v = \sqrt{2gx}$, and from this we have $x = \frac{v^2}{2g}$, as we found before for the height due to the velocity $v$.

## CURVILINEAR MOTION.

Since a body when acted on by no force, or by balanced forces, will move uniformly, and in a straight line, any deviation, either from a uniform speed, or from a rectilinear direction, must be due to the action of one or more forces.

Suppose, first, two uniform motions to be imparted to a body, the resulting motion would be rectilinear and uniform, but differing in direction from both. This would be the case if to a body already moving uniformly, we were to apply suddenly an impulsive force in a different direction from that of the body's motion.

A body, on the other hand, with a uniform motion impressed on it, might be acted on by a force in a different direction acting constantly. This constantly acting force might be of the same magnitude at all points of the body's path, or of varying magnitude; it might act in parallel lines, or in lines converging to a point, or otherwise. In any of these cases, and also when there is more than one force acting on the body, the resulting motion is curvilinear, and the velocity generally variable.

## UNRESISTED PROJECTILE.

In the case of an unresisted projectile (*i.e.*, leaving out of account the resistance of the air) we have a body thrown in a certain direction, and thus impressed with a uniform motion in that direction, and acted on constantly by the force of gravity, a force constant in magnitude, and acting in parallel lines.

Referring to Fig. 232 of Rankine, the velocity, given to the body in the direction OA, which we shall call $v_0$, may be considered as made up of the two components, $v_0 \cos \theta$ and $v_0 \sin \theta$, acting in the directions O$x$ and O$z$, respectively. The body

body has thus in a horizontal direction a uniform motion; hence no force acts on it in this direction; and the equations of motion become

$$m\frac{d^2x}{dt^2}=0,\quad m\frac{d^2z}{dt^2}=-mg,\ \text{or}\ \frac{d^2x}{dt^2}=0,\ \text{and}\ \frac{d^2z}{dt^2}=-g.$$

Integrating, and remembering that when $t=0$ the horizontal and vertical velocities are respectively $v_0\cos\theta$ and $v_0\sin\theta$, we have

$$\frac{dx}{dt}-v_0\cos\theta=0,\ \text{and}\ \frac{dz}{dt}-v_0\sin\theta=-gt,$$

$$\therefore \frac{dx}{dt}=v_0\cos\theta,\ \text{and}\ \frac{dz}{dt}=v_0\sin\theta-gt.$$

Integrating again, and remembering that for $t=0$, $x$ and $z$ are 0, we have

$$x=v_0\cos\theta\cdot t,\ \text{and}\ z=v_0\sin\theta\cdot t-\frac{gt^2}{2}.$$

But in order to determine the form of the curvilinear path we must eliminate $t$ from these equations, and we obtain

$$z=x\tan\theta-\frac{gx^2}{2v_0^2\cos^2\theta};$$

hence the path is a parabola.

The velocity at any instant is the resultant of the horizontal and vertical velocities, respectively.

$$\therefore v^2=v_0^2-2gz\ \therefore z=\frac{v_0^2-v^2}{2g}=\frac{v_0^2}{2g}-\frac{v^2}{2g}.$$

Thus the projectile (if thrown upward) at first rises to a height $\frac{v_0^2}{2g}$, when it loses all its velocity and begins to descend, and when it has reached the same height as before its velocity is again $v_0$, as we can see when $z=0$, $v^2-v_0^2=0$, or $v=v_0$.

EXAMPLES.

1. An inclined plane 10 feet long is inclined to the horizon at an angle of 30°, find the time which a body will take to move from the top to the bottom, and the velocity it will have acquired when it arrives there (being acted on by gravity alone).

*Solution.* $AB = \frac{1}{2}g \sin\theta\, t^2 \therefore 20 = g \sin\theta\, t^2$. (Fig. 6.)

$$\therefore t^2 = \frac{20}{\frac{1}{2}(32)} = \frac{20}{16} \therefore t = \frac{\sqrt{20}}{4} = \frac{4.472}{4} = 1.118.$$

$$v = \sqrt{2g\text{AB}\sin\theta} = \sqrt{2gh} = \sqrt{320} = 4\sqrt{20} = 17.888.$$

2. Given in the following right-angled triangle (Fig. 7), AB = 10, BAC = 30°. Find the time a body would take when acted on by gravity alone, to fall through each of the sides respectively (AB being vertical).

3. A body acted on by gravity is constrained to move on the arc of a circle (Fig. 8) from A to C, radius 10; find the time of describing the arc (quadrant) and the velocity acquired when it reaches C.

4. An unresisted projectile is thrown upwards at an angle of 45° to the horizon, from a point 10 feet above the surface of the earth, with a velocity of 12 feet per second, find the equation of its path, and the time it will take to reach the earth again.

*Solution.* Assuming the origin at the starting point, equation of path is

$$z = x \tan\theta - \frac{gx^2}{2v_0^2 \cos^2\theta}, \text{ or}$$

$$z = x\frac{1}{\sqrt{3}} - \frac{32x^2}{2(12)^2\frac{1}{2}},\ x = \frac{x}{\sqrt{3}} - \frac{32x^2}{144}, \text{ or } z = \frac{x}{\sqrt{3}} - \frac{2x^2}{9}.$$

We have also $z = v_0 \sin\theta\, t - \frac{gt^2}{2}$; when $z = -10$,

$$-10 = 12\sqrt{\tfrac{1}{2}}\cdot t - 16t^2 \therefore 16t^2 - 6\sqrt{2}\cdot t = 10 \therefore t^2 - \frac{3\sqrt{2}}{8}t = \tfrac{5}{8}.$$

$$\therefore t = \frac{3\sqrt{2}}{16} \pm \sqrt{\frac{18}{256} + \frac{160}{256}} = \frac{3\sqrt{2} \pm \sqrt{178}}{16}.$$

5. An unresisted projectile is thrown upwards at an angle of 37° from the surface of the earth, find the time when it will reach the earth, and the velocity it will acquire when it again reaches the earth, the velocity of throwing being 30 feet per second.

The student should now read carefully Rankine's Art. 537, and below I shall give the work, viz., the equation of the path of an unresisted projectile is, as we have seen,

$$z = \tan\theta \cdot x - \frac{gx^2}{2v_0^2\cos^2\theta}, \text{ also } \frac{dz}{dt} = v_0 \sin\theta - gt.$$

$$\frac{dx}{dt} = v_0\cos\theta \therefore \frac{dz}{dx} = \frac{v_0\sin\theta - gt}{v_0\cos\theta} \therefore 1 + \left(\frac{dz}{dx}\right)^2 = 1 + \frac{(v_0\sin\theta - gt)^2}{v_0^2\cos^2\theta}$$

$$= \frac{v_0^2\cos^2\theta + v_0^2\sin^2\theta - 2v_0\sin\theta\, gt + g^2t^2}{v_0^2\cos^2\theta}$$

$$= \frac{v_0^2 - 2v_0\sin\theta\, gt + g^2t^2}{v_0^2\cos^2\theta} = \frac{v^2}{v_0^2\cos^2\theta}, \text{ by Rankine, Art. 534.}$$

$$\therefore Q = W \div \sqrt{1 + \left(\frac{z}{dx}\right)^2} = W \div \frac{v}{v_0\cos\theta} = \frac{Wv_0\cos\theta}{v}.$$

$$\text{Again, } \frac{dz}{dx} = \tan\theta - \frac{gx}{v_0^2\cos^2\theta} \therefore \frac{d^2z}{dx^2} = \frac{-g}{v_0^2\cos^2\theta}$$

$$\therefore v = \left\{1 + \frac{dz^2}{dx^2}\right\}^{\frac{3}{2}} \div \frac{d^2z}{dx^2} = \left(\frac{v}{v_0\cos\theta}\right)^3 \cdot \frac{v_0^2\cos^2\theta}{g}.$$

## REVOLVING SIMPLE PENDULUM. (Rankine, Art. 539.)

The only forces acting on the mass at A (Fig. 9) are the tension on the string P, the centrifugal force Q, and the weight of the mass at W. The first two give a resultant equal and opposite to the weight of the mass at A, as seen in the figure; moreover, the triangle ADE is similar to CAB, and hence we have

$$\frac{BC}{AB} = \frac{AE}{ED} = \frac{AE}{Q} \therefore \frac{h}{r} = \frac{W}{Q}; \text{ but } Q = \frac{Wv^2}{gr}.$$

$$\therefore \frac{h}{r} = \frac{gr}{v^2} \therefore hv^2 = gr^2.$$

Now when $n$ is the number of turns per second, $v = 2\pi nr$, and hence $4\pi^2n^2r^2h = gr^2 \therefore h = \frac{g}{4\pi^2n^2}$.

### EXAMPLES.

1. Given $n = 4, 3, 7, 5, 8$, find $h$ in each case.
2. Given $h = 4, 7, 30, 29$ inches respectively, find $n$.

## DEVIATING FORCE.

We have already seen that a body acted on by no force, or by balanced forces, moves uniformly and in a straight line, and conversely; hence if a body describes a curvilinear path, that body must be acted on by some force, and when the body describes a circular path with a uniform velocity, it is evident that the force acting on it, which causes it to change its direction from that of the tangent, is a force which has no effect in changing its velocity in the circular path, hence it can have no component in the direction of the tangent, and hence must be wholly normal to the path. It is then called a Deviating Force, as its only effect is to turn the body from its rectilinear course without affecting its velocity. This is equal and opposite to what is called the centrifugal force, and from what we have found before, its magnitude is

$$Q = \frac{Wv^2}{gr} = \frac{Wa^2r^2}{gr} = \frac{Wa^2r}{g}. \quad \text{Rankine, Art. 540.}$$

## COMPONENTS OF THE DEVIATING FORCE.

Since the deviating force Q acts in a direction normal to the body's path at any point, it may be represented by a line AL in the direction AO (Fig. 10), and this may be decomposed into two components, one AC, in the direction AD, and the other AK, in the direction AB, and since the triangle ACL is similar to ADO, we have,

$$AL : AC : CL = AO : AD : OD, \text{ or } Q : Q_x : Q_y = r : x : y.$$

$$\therefore Q_x = \frac{Qx}{r}, \text{ and } Q_y = \frac{Qy}{r}; \text{ but } Q = -\frac{Wa^2r}{g}, \therefore \text{we have}$$

$$Q_x = -\frac{Wa^2x}{g}; \; Q_y = -\frac{Wa^2y}{g}.$$

The negative signs are given because, considering the centrifugal force as positive, a force directly opposite to it should be accounted negative.

Rankine's second demonstration needs no comment.

## STRAIGHT OSCILLATION.

Straight Oscillation consists in the oscillation of a body backwards and forwards in a straight line, Fig. 11. Imagine the body starting at A and moving toward B, acted on by some force which draws it towards the central point C.

When it arrives at C the velocity already communicated to it by the action of the force causes it to go on farther, and it would continue to reach farther and farther from C were it not that as soon as it leaves C the force comes again into play, tending to draw it towards C, and hence diminishing its velocity, till when it arrives at B it has lost it all, and now the force causes it to return towards C, and thence it goes to A, etc., etc.

This force, as a general case, is proportional to its distance from C, that is, at C it is 0, and it has its greatest value at the points A and B, viz., when the body is farthest from C; or it is equal in every case to $x$ multiplied by a constant, $x$ being the distance from C.

Now let us imagine one body moving uniformly in the circumference of a circle, which is caused, as we have seen, by two deviating forces, $-\frac{Wa^2x}{g}$ and $-\frac{Wa^2y}{g}$, at right angles to each other, and then suppose another body to be acted on by the force $-\frac{Wa^2x}{g}$ alone; imagine both to start together at E, Fig. 12, while the first describes the arc EA, the second would describe EB, for the equation of motion of the second being $m\frac{d^2x}{dt^2} = -\frac{W}{g}a^2x$, we should have,

$$\frac{d^2x}{dt^2} = -a^2x = -a^2r \cos at. \therefore \frac{dx}{dt} = -ar \sin at. \therefore x = r \cos at.$$

This force, $\frac{Wa^2x}{g}$, has a different value for every different position of the body; at O it is 0, and it has its greatest value when $x$

is greatest, or when $x = OE = r$; hence the greatest value of this force is $Q = \frac{Wa^2r}{g}$.

For the body revolving in the circle we have,

$$a = 2\pi n \therefore \frac{1}{n} = \frac{2\pi}{a}; \text{ but } a = \sqrt{\frac{gQ}{rW}} \therefore \frac{1}{n} = 2\pi\sqrt{\frac{rW}{gQ}}.$$

## ELLIPTICAL OSCILLATIONS.

This is given in Art. 543 of Rankine. To derive equations (4), I would refer the student to Straight Oscillation.

## SIMPLE CIRCULAR PENDULUM.

To find the time occupied in a vibration of the simple circular pendulum, we may, as Rankine says, regard the oscillations when the arc is small, as oscillation in a straight line, and then we have, as he deduces, $\frac{1}{n} = 2\pi\sqrt{\frac{l}{g}}$ for the time of a double oscillation, where the quantity $n$ represents the number of turns per second. $\therefore \frac{1}{n}$ is the part of a second, or the number of seconds occupied in one double oscillation, that is, from the time it leaves one extreme point till it returns there. But if the arc of vibration is not very small, it becomes necessary to obtain a more accurate formula.

To do this we have merely to refer to the formula developed under the head of Constrained Motion, for the time down any curve, and apply it to the present case, viz., $t = 2\int\frac{ds}{\sqrt{2g(h-x)}}$ where $t$ represents the time of a single oscillation, viz., the time of passing from A to E, Fig. 13, and where the line DC is taken as axis of $x$, D being the origin, letting $AC = l$, $BD = h$. From the equation of the circle, $y^2 = 2lx - x^2$, we have,

$$\frac{dy}{dx} = \frac{l-x}{y} \therefore \frac{ds}{dx} = \frac{l}{y} = \frac{l}{\sqrt{2lx-x^2}}.$$

$$\therefore t = 2\int_0^h \frac{l dx}{\sqrt{2lx - x^2}\sqrt{2g(h-x)}} =$$

$$\frac{2l}{\sqrt{2g}}\int_0^h \frac{dx}{\sqrt{h-x}\sqrt{2l-x}\sqrt{x}} = \frac{2l}{\sqrt{2g}}\int_0^h \frac{dx}{\sqrt{hx-x^2}\sqrt{2l-x)}}$$

$$= \frac{2l}{\sqrt{4gl}}\int_0^h \frac{dx}{\sqrt{hx-x^2}}\left(1-\frac{x}{2l}\right)^{-\frac{1}{2}} = \sqrt{\frac{l}{g}}\int_0^h \frac{dx}{\sqrt{hx-x^2}}\left(1-\frac{x}{2l}\right)^{-\frac{1}{2}}.$$

This cannot be integrated except approximately.

Expanding $\left(1-\frac{x}{2l}\right)^{-\frac{1}{2}}$ we have $\left(1+\frac{x}{4l}+\frac{3x^2}{8l^2}, \text{etc.}\right)$

$$\therefore t = \sqrt{\frac{l}{g}}\int_0^h \left(1+\frac{x}{4l}+\frac{3x^2}{8l^2}+\text{etc.} \ldots\right)\frac{dx}{\sqrt{hx-x^2}}.$$

The greatest value of $x$ is $h$, and if $h$ is so small that we may omit $\frac{x}{4l}$ and all the higher powers, we shall have for an approximate formula,

$$t = \sqrt{\frac{l}{g}}\int_0^h \frac{dx}{\sqrt{hx-x^2}} = \sqrt{\frac{l}{g}}\left\{\text{versin}^{-1}\frac{2x}{h}\right\}_0^h = \pi\sqrt{\frac{l}{g}}$$

the same value as was found before for a single oscillation.

If, however, the value of $h$, as compared with $l$, is too large to render it safe to omit $\frac{x}{4l}$, but we can omit the higher powers of $\frac{x}{l}$, we shall have,

$$t = \sqrt{\frac{l}{g}}\left\{\text{versin}^{-1}\frac{2x}{h} + \frac{1}{4l}\int\frac{x dx}{\sqrt{hx-x^2}}\right\}_0^h$$

$$= \sqrt{\frac{l}{g}}\left\{\text{versin}^{-1}\frac{2x}{h} + \frac{1}{4l}\left(\frac{h}{2}\text{versin}^{-1}\frac{2x}{h} + \sqrt{hx-x^2}\right)_0^h\right\}$$

$= \pi\sqrt{\frac{l}{g}}\left(1+\frac{h}{8l}\right)$, which is a nearer approximation.

Take, however, the formula, $t = \pi\sqrt{\frac{l}{g}}$, we can thus determine the value of $t$ when $l$ is given, or of $l$ when $t$ is given.

### EXAMPLE.

1. Find the length of the simple circular pendulum which is to beat seconds at a place where $g = 32\frac{1}{6}$.

*Solution* $t = \pi\sqrt{\frac{l}{g}} \therefore t^2 = \pi^2\frac{l}{g} \therefore l = \frac{tg}{\pi^2} \therefore$

$$l = \frac{32\frac{1}{6}}{(3.1416)^2} = 3.2591 \text{ feet.}$$

2. What is the time of vibration of a simple circular pendulum 5 feet long?

3. What is the time of vibration of a simple circular pendulum 5 inches long?

4. How long must a simple circular pendulum be which is to beat once in $\frac{1}{2}$ a second? in $\frac{1}{3}$ second? in 3 seconds? in 5 seconds? in 10 seconds?

## SIMPLE CYCLOIDAL PENDULUM.

The equation of the cycloid is $y = a \text{ versin}^{-1}\frac{x}{a} + (2ax - x^2)^{\frac{1}{2}}$.

$$dy = \frac{adx}{\sqrt{2ax - x^2}} + \frac{(a - x)dx}{\sqrt{2ax - x^2}} = \frac{(2a - x)dx}{\sqrt{2ax - x^2}} = \sqrt{\frac{2a - x}{x}}dx$$

$$\therefore \frac{dy^2}{dx^2} = \frac{2a - x}{x} \therefore \frac{ds^2}{dx^2} = \frac{2a}{x} \therefore ds = \left(\frac{2a}{x}\right)^{\frac{1}{2}}dx;$$

hence the general formula, $t = 2\int\frac{ds}{\sqrt{2g(h - x)}}$, becomes

$$t = 2\frac{\sqrt{2a}}{\sqrt{2g}}\int_0^h\frac{dx}{\sqrt{hx - x^2}} = \left(\frac{a}{g}\right)^{\frac{1}{2}}\left\{\text{versin}^{-1}\frac{2x}{h}\right\}_0^h = \pi\sqrt{\frac{a}{g}}.$$

The expression is independent of $h$, so that the time of vibration is the same whether the arc be large or small.

A body can be made to vibrate in a cycloidal arc by suspending it by a flexible string between two cycloidal cheeks (Fig. 14); this is shown from the fact that the evolute of the cycloid is another

cycloid. To prove this we have, from the equation of the cycloid,

$$y = a \text{ versin}^{-1}\frac{x}{a} + (2ax - x^2)^{\frac{1}{2}}, \ \frac{dy}{dx} = \sqrt{\frac{2a - x}{x}},$$

$$\frac{ds}{dx} = \sqrt{\frac{2a}{x}}, \ \frac{d^2y}{dx^2} = -\frac{a}{x^{\frac{3}{2}}\sqrt{2a - x}} \ \therefore \text{ the radius of curvature,}$$

$$\rho = \frac{\left(\frac{ds}{dx}\right)^3}{-\frac{d^2y}{dx^2}} = \frac{\frac{(2a)^{\frac{3}{2}}}{x^{\frac{3}{2}}}}{\frac{a}{x^{\frac{3}{2}}\sqrt{2a - x}}} = \frac{(2)^{\frac{3}{2}}a^{\frac{1}{2}}}{\sqrt{2a - x}} = \frac{2(2a)^{\frac{1}{2}}}{\sqrt{2a - x}}.$$

And we have for the evolute, the relation

$$ds' = d\rho \ \therefore s' = \int_0^{2a} d\rho,$$

$s'$ being the length of the arc of the evolute, and $\rho$ the radius of curvature of the original curve. $2a$ and $0$ are the limits of $x$; and observing that when $x = 2a$, $\rho = 0$, we have,

$$s' = \rho \ \therefore s' = 2(2a)^{\frac{1}{2}}\sqrt{2a - x},$$

or if we transform coördinates to B, we have

$$s' = 2(2ax)^{\frac{1}{2}} \ \therefore \ s'^2 = 8ax,$$

which is the equation of another cycloid just equal to the first.

This motion along a vertical cycloid may also be obtained by letting a body move along a groove in the form of a cycloid acted on by gravity alone, and in this case the time of descent of the body to the lowest point O. is precisely the same whether the body is placed either at A. B. C. or any other point of the curve. When the body has reached the lowest point, it has of course acquired sufficient velocity to cause it to ascend on the other part of the curve to a height equal to that from which it started.

The converse of the above is true, viz., that a body in pursuing a path such that the time of descent from any position to the lowest point shall be the same, must pursue the arc of a cycloid.

The force acting on the body at any point is the force of gravity acting vertically; this can be resolved into a tangential and a nor-

mal component, the normal component is resisted by the tension of the string, and the tangential force is $g \cos \theta$, where $\theta$ is the angle between the tangent to the curve and the axis of $x$,

$$\therefore \mathrm{T} = g \cos \theta = g\frac{dx}{\sqrt{dx^2 + dy^2}} = g\frac{dx}{ds}.$$

Now the tangential force can be shown by the Calculus of Variations to vary as the length of the arc from the lowest point, $\mathrm{T} = \mathrm{K}s$, K being a constant,

$$\therefore \mathrm{K}s = g\frac{dx}{ds} \therefore \mathrm{K}sds = gdx \therefore \frac{\mathrm{K}s^2}{2} = gx \therefore s^2 = \frac{2g}{\mathrm{K}}x, \text{ or } s^2 = ax,$$

which is the equation of the cycloid.

## RESIDUAL FORCES.

Rankine, in Art. 546, speaks of Residual Forces; the result of his reasoning is to show that the weight of a body is not absolutely a measure of the force of gravity that acts on it, for the body is acted on first by the force of gravity tending to draw it towards the earth, and secondly, by the centrifugal force which it has in consequence of its rotation with the earth around the centre of the earth, which acts contrary to its weight, and hence the weight of a body is really the difference between these two.

## WORK.

All forces act for a certain amount of time; thus in driving a nail with a hammer, there must have been a time when the nail was half way in, etc., and hence whenever a force produces motion instead of pressure, it continues to act on the body for a certain time, and through a certain space.

If a resistance is overcome through a certain space, we call it so much *Work* done. Thus if 5 pounds is to be raised through a height of 3 feet, a certain amount of force is required, and a certain amount of work is to be done.

It is plain that to raise a 5 pound weight through a height of 3 feet requires that 3 times as much work should be done as if it

were to be raised through a height of 1 foot. So, likewise, in the same case, 5 times as much work is done as would be in raising a 1 pound weight 3 feet.

The most common unit used in measuring Work is that which is performed in raising one pound through a height of one foot, and is called a *foot pound.*

In the case of raising a weight, the resistance overcome is gravity, but work may be done against other resistances, as, for instance, the work of a locomotive in moving a train along a level track ; here the resistance is friction, and also the resistance of the air. A horse dragging a carriage along the ground, all kinds of machinery moved by steam or water, or any other motive power, mills, etc., all require work to be performed to make them fulfill their respective functions.

EXAMPLES.

1. The ram of a pile driver weighs 1200 lbs., and has a fall of 15 feet, how much work is performed in raising it?

*Solution.* Work performed $= 1200 \times 15 = 18{,}000$ foot pounds.

2. How much work is performed by a steam engine in raising 12,000 lbs. of water to a reservoir situated 30 feet above the free upper level?

3. How much work is performed in raising a weight of 550 lbs. through a height of 35 feet?

## WORK WITH REFERENCE TO TIME.

Thus far we have considered work independently of the time consumed in performing it, but in machinery we must, in most cases, take into account the time; thus one machine may be capable of performing 4500 foot pounds of work per minute, and another can perform the same in $\frac{1}{2}$ that time. The amount of work generally taken as a unit in machines is what is known as a *horse power*, and is equal to 33,000 foot pounds per minute; hence,

a steam engine capable of raising 16,500 pounds of ore from a mine 40 feet deep in 5 minutes, would in one minute raise 3,300 lbs., and would hence perform $3,300 \times 40 = 4(33,000)$ foot pounds of work per minute, and hence it is an engine of 4 horse power.

EXAMPLES.

1. How many horse power would be required to supply a city of 5,000 inhabitants with water, each one using 10 gallons of water per day, the engine working 6 hours per day, and the water being raised through a height of 100 feet? Weight of water = 62 lbs. per cubic foot.

2. Required the horse power of an engine which is to give power to 5 tilt hammers, each of which weighs 500 lbs., and makes 50 lifts per minute, the perpendicular height of each lift being 2 feet?

3. How many hours per day would it be necessary to work an engine of 36 horse power to supply a city of 35,000 inhabitants with water, each inhabitant using 6 gallons, the water being lifted through a height of 25 feet?

4. How many gallons of water can be pumped from a mine by a 10 horse power engine in 11 hours, the depth of the mine being 100 feet?

5. What is the rate in miles per hour of a train of cars whose weight is 70 tons, the friction being 7 lbs. to the ton, drawn by an engine of 50 horse power?

We have seen by the first law of motion that if a body is in motion it will continue to move in the same direction, and with the same speed, unless and until it is acted on by some external force. We have also seen that, strictly speaking, an impulsive force does not occur in nature, but that every force acts for some definite time, and hence that when it produces motion it acts through a certain space.

From these facts we conclude, 1st, that a force when unresisted produces change of velocity, since uniform motion is a condition

of equilibrium of a body. 2d, suppose a force to act on a body while the body moves through a certain space, it produces a certain velocity which the body retains until it is opposed by some external force (resistance), and to destroy this velocity an equal amount of *work* must be done to what was done in imparting it to the body; hence a body which has had a certain amount of work impressed upon it is, in its turn, capable of performing that amount of work on another body.

Again, suppose the body to overcome a resistance which requires a less amount of work than the amount impressed on the body, then the body will continue to move with a less velocity, and will still be capable of performing an amount of work equal to the difference between the amount impressed on it, and the amount of work it has performed.

This, in other words, is that the energy exerted (*i. e.*, the amount of energy lost) is equal to the work performed. This is called the principle of Conservation of Energy, and shows us that a machine is not capable of generating, but only of transferring energy.

Rankine proves this algebraically in Art. 518.

This principle of the Conservation of Energy is very well illustrated in the hydraulic press (of which Fig. 17 represents a section), where the product of the power by the distance through which it moves is equal to the product of the weight by the distance through which it is raised.

Energy is, then, Capability for performing Work.

Energy is of two kinds, Actual and Potential.

To illustrate the two, imagine the ram of a pile driver weighing 1 ton, and having a fall of 15 feet, to have reached a point 5 feet from its point of starting (the top). If it were now to meet the head of the pile it would exert on it $2{,}000 \times 5 = 10{,}000$ foot pounds; this is its *Actual Energy*, but it has still 10 feet more to fall, and if allowed to fall these remaining 10 feet before coming in contact with the head of the pile, it would exert on it $2{,}000 \times 10 = 20{,}000$ foot pounds more.

This latter is its *Potential Energy.*

Thus Potential Energy is the force multiplied by the distance through which its point of application is capable of being moved.

If the force varies at different points of the path, we must multiply each different value of the force by the distance through which its point of application is moved. If it varies at every point we then pass to the limit, and we have $\int P ds$.

EXAMPLES.

1. Given a force $P = ax + bx^3$, find the amount of work done by this force acting through a distance, $x = a$, from the origin.

2. Given $P = \sqrt{r^2 - x^2}$, find the amount of work done while its point of application moves through a distance $r$ from O, on the axis $Ox$.

## CONSERVATION OF ENERGY.

The principle of the Conservation of Energy just enunciated and proved is of the greatest importance in machinery. To fix our thoughts, let us suppose we have a 4 horse power steam engine; the force developed by the engine would be capable of doing $4(33,000) = 132,000$ foot pounds of work per minute; this work causes the first wheel to move, and since this wheel has impressed upon it a force of 4 horse power, it is itself capable of exerting a force of 4 horse power, and this it transfers to its neighbor, that to the next, and so on; hence we should expect from the final working piece of the machine an amount of work equal to 4 horse power, and this would be the case were it not for the friction, the weight of the different parts that have to be lifted, and other resistances which require that a certain amount of work should be done to overcome them; hence we never get practically from a machine as much effective work as we impress upon it.

What has been said shows that energy is never really lost, but can merely be transferred from one body to another; hence a body on which work has been performed has so much energy stored, and is in its turn capable of performing that amount of work.

## VIRTUAL VELOCITIES.

The principle of the conservation of energy may be (and is) used to determine the conditions of equilibrium of forces applied to any connected system of points; it is then known as the principle of Virtual Velocities.

This principle of virtual velocities could be used to demonstrate the parallelogram of forces, and hence we could build on it the whole of Statics.

To apply it we imagine the system to move in any way consistent with the equilibrium of the system. Thus take, as an example, a lever of the 1st order, *i. e.*, a bar used to raise a weight. Suppose the lever, Fig. 18. to be at first in the position AE, the power P being applied at A, and the weight at E, when the lever has been caused to take the position BD, the power P has acted through a height AB $\therefore$ P·AB is the work done in a vertical direction, the weight W has been raised through a height DE; hence the work done in overcoming the weight is W·DE; hence

$$P \cdot AB = W \cdot DE \therefore P = W\frac{DE}{AB}, \text{but} \frac{DE}{AB} = \frac{CE}{AC} \therefore P = W\frac{CE}{AC}.$$

## PARALLELOGRAM OF FORCES.

We will now prove the proposition of the parallelogram of forces by means of virtual velocities.

*Definition.* Suppose the point of application A of the force P, Fig. 19, to suffer a small displacement by which it is moved to A′, the force P keeping parallel to its former position. If from A′ we drop a perpendicular on the first direction of the force, the distance $Am$ is called the virtual velocity of the force P, and hence, virtual velocity of $P = Am = AA' \cos \theta$.

If a set of forces are in equilibrium, the total amount of work done by them must be 0, or $\Sigma Px = 0$. Consider three forces, P, Q, and R, Fig. 20, originally applied at A, which point has moved to A′, we have,

$$P \cdot Am + QAn + RAp = 0.$$

Let PAR $= \beta$, QAR $= \alpha$, PAQ $= \gamma$; also A′AP $= \theta$; then P·AA′ $\cos\theta$ + Q·AA′ $\cos(\theta + \gamma)$ + R·AA′ $\cos(\beta - \theta) = 0$: or $P \cos\theta + Q \cos\theta \cos\gamma - Q \sin\theta \sin\gamma + R \cos\beta \cos\theta + R \sin\beta \sin\theta = 0$; or

$(P + Q \cos\gamma + R \cos\beta)\cos\theta + (R \sin\beta - Q \sin\gamma)\sin\theta = 0$; but since $\theta$ is arbitrary, by the principle of Indeterminate Coefficient, $P + Q \cos\gamma + R \cos\beta = 0$, and $R \sin\beta - Q \sin\gamma = 0$.

$\therefore R \cos\beta = -(P + Q \cos\gamma)$. (1.) $R \sin\beta = Q \sin\gamma$. (2.)

$$\therefore \frac{R}{\sin\gamma} = \frac{Q}{\sin\beta} = \frac{P}{\sin\alpha},$$

and by squaring and adding (1.) and (2.),

$$R^2 = P^2 + Q^2 + 2PQ \cos\gamma,$$

which proves the proposition of the parallelogram of forces.

## APPLICATIONS OF THE PRINCIPLE OF VIRTUAL VELOCITIES.

### THE SCREW.

To find the relation between the power and weight in the screw. Let $h$ = the pitch of the screw, Fig. 21, and suppose the power applied at B, the work done by the power in one revolution, is $P \cdot 2\pi r$; the weight will thus be raised through a height $h$; hence the work done is also $= Wh$ $\therefore$

$$P \cdot 2\pi r = Wh \therefore P = W\frac{h}{2\pi r}.$$

### THE WHEEL AND AXLE.

Let $a$ be the radius of the wheel on which the power acts, and $b$ that of the axle on which the weight acts. In one revolution the power moves through $2\pi a$, and the weight through $2\pi b$.

$$\therefore P \cdot 2\pi a = W \cdot 2\pi b \therefore Pa = Wb \therefore P = W\frac{b}{a}.$$

Note on Art. 520, Rankine. The student must take notice that the force F is a force acting on the body while it has the given motion; then F $\cos\alpha$ denotes the component of the force in the

direction $Ox$, and $\Delta s \cos \alpha'$ the component of the motion, so that the product of these two denotes the work done by the force in the direction $Ox$.

Note on Art. 524.

To make it plain, let us take three bodies (Fig. 22) of mass, $m$, $m_1$ and $m_2$, respectively, at distances $x$, $x_1$ and $x_2$, from $Oy$. Suppose in one second they move as in the figure, so that $x'$, $x_1'$ and $x_2'$, are their distances from $Oy$ at the end of the motion, we have then, $x_0$ being the distance of their centre of gravity from $Oy$, the formulæ,

$$x_0 = \frac{mx + m_1x_1 + m_2x_2}{m + m_1 + m_2} = \frac{\Sigma mx}{\Sigma m},$$

$$x'_0 = \frac{mx' + m_1x_1' + m_2x_2'}{m + m_1 + m_2} = \frac{\Sigma mx'}{\Sigma m},$$

$$\therefore x_0' - x_0 = \frac{m(x' - x) + m_1(x_1' - x_2) + m_2(x_2' - x_2)}{m + m_1 + m_2} = \frac{\Sigma m(x' - x)}{\Sigma m};$$

but $x' - x = u$, $x_1' - x_1 = u_1$, $x_2' - x_2 = u_2$, $x_0' - x_0 = u_0$.

$$\therefore u_0 = \frac{\Sigma mu}{\Sigma m} \therefore u_0 \Sigma m = \Sigma mu.$$

So also $v_0 \Sigma m = \Sigma mv$, and $w_0 \Sigma m = \Sigma mw$.

Hence if we can ascertain the motion of a body's centre of gravity, and we know its total mass, we can ascertain its total momentum.

In connection with what precedes, Rankine, Chapter II., through Art. 528, should be studied.

## ACTUAL ENERGY.

If a body, whose weight is W, fall through a height $h$, the force of gravity has impressed on it an energy W$h$, which it is in its turn capable of exerting on another body; thus if it were to fall into the pan A, Fig. 23, suspended from a string passing over a wheel, at the other end of which is another weight W, the impetus

of its fall would cause the second weight to be lifted through a height $h$. Now in falling through a height $h$, it acquires a velocity $v = \sqrt{2gh} \therefore h = \frac{v^2}{2g}$; hence the actual energy of the falling body is $Wh = \frac{Wv^2}{2g} = \frac{mv^2}{2}$.

Now it matters not whether the direction of its motion is vertical or not, if it have a velocity $v$ imparted to it. The amount of energy impressed on it is $\frac{mv^2}{2}$, and hence its actual energy is $\frac{mv^2}{2}$ $= \frac{1}{2}$ its mass multiplied by the square of the velocity it has acquired.

## COMPONENTS OF ACTUAL ENERGY.

Since a body's actual energy is $\frac{mv^2}{2}$, in whatever direction the body is moving, whether the path be straight or curved when the body's velocity is $v$, if we decompose the velocity into 3 components, in the directions $Ox$, $Oy$ and $Oz$, respectively, these components will be $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, respectively (see my notes, p. 8); hence since the velocity in the direction $Ox$ is $\frac{dx}{dt}$, the body's actual energy in that direction is $\frac{m}{2}\left(\frac{dx}{dt}\right)^2$, so in the direction $Oy$ it is $\frac{m}{2}\left(\frac{dy}{dt}\right)^2$, and in $Oz$, $\frac{m}{2}\left(\frac{dz}{dt}\right)^2$, and since $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2 = v^2$ $\therefore \frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{m}{2}\left(\frac{dy}{dt}\right)^2 + \frac{m}{2}\left(\frac{dz}{dt}\right)^2 = \frac{mv^2}{2}$, or the actual energy of the body is equal to the sum of the components of the actual energy, in the directions $Ox$, $Oy$ and $Oz$.

## RELATION OF ACTUAL AND POTENTIAL ENERGY.

### Energy Stored and Restored.

If a body have a velocity $v$, its actual energy is, as we have seen, $\frac{Wv^2}{2g} = Wh$, and if it then move to a point where its velocity is $v'$, its actual energy becomes $\frac{Wv'^2}{2g} = Wh'$ ($h$ and $h'$ being respectively the heights due to the velocities $v$ and $v'$); hence the increase of actual energy when $v' > v$ is $\frac{W}{2g}(v'^2 - v^2)$, or the loss of actual energy when $v' < v$ is $\frac{W}{2g}(v^2 - v'^2)$. The 1st $\frac{W}{2g}(v'^2 - v^2) = W(h' - h)$, and the 2d, $\frac{W}{2g}(v^2 - v'^2) = W(h - h')$, and this must be equal to the product of the force acting on it by the distance through which its point of application has been moved, or in other words, to the Potential Energy exerted.

If, as in the first case, Potential Energy is changed into Actual Energy, the velocity being increased, the amount of actual energy thus gained is said to be stored, and the body is able to do so much more work, but in the second case, when Actual Energy is converted back into Potential Energy, it is said to be restored.

## DYNAMICAL MEASURE OF FORCES.

It was for a long time a subject of dispute whether a force was to be measured by (was proportional to) its momentum ($mv$), or by the actual energy it had imparted to a body, some contending for the first, saying that to impart to a body a double or treble momentum required a double or treble force, and the advocates of the latter contending that to impart a double or treble energy required a double or treble force.

The misunderstanding arose from the fact that the disputants did not examine whether they referred their forces to equal times

or equal spaces, and if this consideration be made we shall find that the two measures are equivalent; for, to impart to a body a double momentum in the same *time* requires a double force, and to give it a double actual energy (power of performing work, as, for instance, to lift a double weight) while moving through the same space, requires twice the force, and indeed

$$\frac{d\left(\frac{mv^2}{2}\right)}{ds} = \frac{d(mv)}{dt}, \text{ for}$$

$$\frac{d\left(\frac{mv^2}{2}\right)}{ds} = mv\frac{dv}{ds}; \text{ but } ds = vdt \therefore \frac{d\left(\frac{mv^2}{2}\right)}{ds} = m\frac{dv}{dt}, \text{ and}$$

$$\frac{d(mv)}{dt} = m\frac{dv}{dt} \therefore \frac{d\left(\frac{mv^2}{2}\right)}{ds} = \frac{d(mv)}{dt};$$

that is, the increase of actual energy per unit of space is equal to the increase of momentum per unit of time.

## ENERGY DUE TO OBLIQUE FORCE.

To illustrate this take the case of a body moving down an inclined plane (Fig. 24). Suppose a body of weight W to start at the top of the plane and slide down, acted on by gravity alone. Here the only force acting is W, and this acts vertically, making with the direction of the body's motion an angle, $\frac{\pi}{2} - \theta$, hence the component of the force in this direction is $W\cos\left(\frac{\pi}{2} - \theta\right) = W\sin\theta$, and this is the only component that is effective in imparting energy to the body (the other component $W\sin\left(\frac{\pi}{2} - \theta\right) = W\cos\theta$, being counteracted by the reaction of the plane), hence the actual energy acquired in moving from B to A is $W\sin\theta\cdot BA$.

Take another view of it; the velocity acquired in falling through

a height BC is $v = \sqrt{(2g \cdot BC)}$, and then the actual energy of the body is

$$\frac{mv^2}{2} = \frac{W}{g}\frac{v^2}{2} = \frac{W}{g}\frac{2g BC}{2} = W \cdot BC = W \sin\theta \cdot BA,$$

the same as before.

Thus if a force F act on a body, and is inclined at an angle $\theta$ to the direction of the body's motion, the component of this force that is effective in producing motion, or in imparting energy to the body, is F cos $\theta$, and the energy imparted by it in moving through a distance $ds$ is F cos $\theta$ $ds$ $\therefore$ the total actual energy imparted by the force will be $\int F \cos\theta\, ds$.

EXAMPLES.

What is the actual energy acquired by a body in falling through the arc of a circle acted on by gravity alone (Fig. 25)?

1*st Solution.* The force acting is W, of which the component effective in generating energy is W cos $\left(\frac{\pi}{2} - \theta\right)$, and

$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{dy}{dx} = -\frac{x}{y} \therefore \cos\left(\frac{\pi}{2} - \theta\right) = \frac{y}{\sqrt{(x^2 + y^2)}},$$

equation of circle being

$$x^2 + y^2 = r^2 \therefore \cos\left(\frac{\pi}{2} - \theta\right) = \frac{y}{r} \therefore W \cos\left(\frac{\pi}{2} - \theta\right)$$

$$= \frac{Wy}{r} = \frac{W}{r}\sqrt{(r^2 - x^2)}$$

$\therefore$ Actual energy $= \int_0^r \frac{W}{r}\sqrt{(r^2 - x^2)}ds$; but $ds = \frac{r}{y}dx$ $\therefore$

$$\int_0^r \frac{W}{r} y ds = \int_0^r W dx = \Big\{ Wx \Big\}_0^r = Wr.$$

2*d Solution.* $v = \sqrt{(2gh)} = \sqrt{(2gr)}$ $\therefore$ Actual energy $=$

$$\frac{W}{g}\frac{v^2}{2} = \frac{W}{g}\frac{2gr}{2} = Wr.$$

2. What is the actual energy acquired by a body in falling through a quadrant of an ellipse (Fig. 26)?

3. What is the actual energy acquired in falling through the same arc when inverted, as in Fig. 27?

4. What is the actual energy acquired in falling through the inverted arc of a semicycloid?

5. What is the actual energy acquired by a body undergoing straight oscillation, as described in Art. 542 of Rankine, when it arrives at the centre O of the force?

*Solution.* $F = \frac{Wa^2x}{g} \therefore \int_0^r F ds = \int_0^r F dx = \int_0^r \frac{Wa^2xdx}{g}$

$$= \frac{Wa^2}{g}\int_0^r xdx = \frac{Wa^2r^2}{2g}.$$

6. A body moving in the line $Ox$, and starting from O, is acted on by a force $F = ax + bx^3$, find its actual energy when it has reached a point at a distance $c$ from O.

## TOTAL ENERGY.

The actual and potential energy of a body, when added together, give its Total Energy, and this remains unchanged as long as the body is acted on by a reciprocating force.

Thus in the case of the ram of the pile driver, described on page , its actual energy at a point 5 feet from its point of starting is 10,000 foot pounds, while its potential energy at the same point with reference to the earth is 20,000 foot pounds; these two added together give its total energy, which is the same at all points of its path.

Rankine, Ex. III, page 504. (Refer to figure on page 493.)

Instead of equation 6, we should have the following,

$$\frac{Wv^2}{2g} + \int_0^x \frac{Q_1xdx}{x_1} = \text{the total energy} = \frac{Wv^2}{2g} + \frac{Q_1x^2}{2x_1};$$

but from 4 and 5, Art. 541, Rankine, $x = r\cos at = x_1 \cos at$, and $v = \frac{dx}{dt} = -ar\sin at = -ax_1 \sin at = -v_0 \sin at$; hence by substitution we have,

$$\frac{Wv^2}{2g} + \int_0^x \frac{Q_1xdx}{x_1} = \frac{Wv_0^2}{2g}\sin^2 at + \frac{Q_1x_1^2\cos^2 at}{2x_1} =$$

$$\frac{Wv_0^2}{2g}\sin^2 at + \frac{Q_1x_1}{2}\cos^2 at; \text{ but } \frac{Wv_0^2}{2g} = \frac{Q_1x_1}{2} \therefore$$

Total energy $=$

$$\frac{Wv^2}{2g} + \frac{Q_1x^2}{2x_1} = \frac{Q_1x_1}{2}(\sin^2 at + \cos^2 at) = \frac{Q_1x_1}{2}.$$

Note on Art. 558, Rankine.

We have seen by the Third Law of Motion that momentum cannot be generated in one body without generating an equal and opposite momentum in another; hence any momentum caused by an action between two bodies of the same system is neutralized by an equal and opposite momentum, generated in the other body of the system $\therefore$ Q. E. D. Art. 559, now follows at once, since by Art. 524 the resultant momentum is $=$ the product of the total mass by the velocity of its centre of gravity.

## ANGULAR MOMENTUM.

To illustrate angular momentum, let us imagine a wheel revolving on an axis (Fig. 28). Suppose a particle of mass $m$, at a distance $l$ from the axis of revolution, to be revolving around this axis with a velocity $v$; then the force acting on this particle (its momentum) is $mv$, and its moment is $mvl = \frac{W}{g}vl$.

This angular momentum is a couple, and hence can be represented by its moment axis, *i. e.*, by a line perpendicular to its plane and equal to this angular momentum.

Again, since $v$ is the velocity (distance passed over in a unit of time), and $l$ the radius of revolution $\therefore vl =$ twice the area of the sector described by this particle $\therefore$ the Angular Momentum is equal to the product of the mass by the double area swept through by its radius vector.

The same proposition is applicable to a body which describes any other curve than a circle, for by taking a sufficiently small portion we may consider it as part of a circle.

## ANGULAR IMPULSE.

By Angular Impulse is meant the moment of the force which imparts the rotary motion to the body about the axis of revolution, and this must be proportional to the amount of angular momentum it imparts, just as we have seen that forces are proportional to the amounts of momentum they impart in the same time $\therefore$

$$Fldt = mldv.$$

The reasoning used to prove Art. 563 is precisely similar to that used for Art. 558.

## ACTUAL ENERGY OF A SYSTEM OF BODIES.

In studying Art. 564 of Rankine, we must bear in mind that all our ideas of motion are relative, and that the velocity of a body in motion depends altogether on what we regard as our fixed point from which we estimate its motion. Thus if the body has a velocity $v$ with reference to an external point, and the centre of gravity of the system to which the body belongs has a velocity $v'$ in reference to the same external point, and in the same direction, its velocity, with reference to the centre of gravity of the system, will be $v - v'$.

The quantity $\frac{1}{2}(u_0^2 + v_0^2 + w_0^2)\Sigma m$ is the external actual energy, or that due to the motion of its centre of gravity. $\frac{1}{2}\Sigma m(u'^2 + v'^2 + w'^2)$ is its internal actual energy, or that due to the velocity of the body with reference to the centre of gravity of the system. The mutual actions of the bodies can only change the internal energy of the system, for the quantities $u_0 = \frac{\Sigma mu}{\Sigma m}$, $v_0 = \frac{\Sigma mv}{\Sigma m}$, $w_0 = \frac{\Sigma mw}{\Sigma m}$, have been shown to remain unaffected by the mutual actions of the bodies (Art. 559, Rankine). This relation can also be derived from Art. 524 of Rankine, for when the centre of gravity of the system is the point of reference, $u_0 = v_0 = w_0 = 0$, and $\therefore \Sigma mu' = 0$, $\Sigma mv' = 0$, and $\Sigma mw' = 0$.

## COLLISION.

Note on Art. 566 of Rankine.

The student should read in connection with Art. 566 of Rankine, his addendum on page 512.

If $m_1$ and $m_2$ be the masses of the bodies, and $u_1$, $u_2$ their velocities before the collision, the two form a system whose centre of gravity has a velocity $u_0 = \frac{\Sigma mu}{\Sigma m} = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}$.

Then the external actual energy is

$$(m_1 + m_2)\frac{u_0^2}{2}.$$

The velocities in reference to the centre of gravity are $u_1 - u_0$ and $u_2 - u_0$ ∴ the internal actual energy is

$$\frac{m_1(u_1 - u_0)^2}{2} + \frac{m_2(u_2 - u_0)^2}{2}.$$

If the energy were consumed in producing internal vibrations and heat, the velocities relative to the centre of gravity would be

$$-(u_1 - u_0) = u_0 - u_1 \text{ and } -(u_2 - u_0) = u_0 - u_2,$$

and hence those relative to the earth would be

$$v_1 = u_0 + (u_0 - u_1) \qquad v_2 = u_0 + (u_0 - u_2) \text{ or}$$
$$v_1 = 2u_0 - u_1 \qquad v_2 = 2u_0 - u_2, \text{ etc., etc.}$$

Note on Art. 567, Rankine.

If $x$, $y$ and $z$ are the rectangular coördinates of a body's position, we have already seen that $\frac{ds}{dt}$ is its velocity, and that $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, are its components in the directions $Ox$, $Oy$ and $Oz$, respectively, hence the components of its momentum are $m\frac{dx}{dt}$, $m\frac{dy}{dt}$, and $m\frac{dz}{dt}$, their rates of variation being $m\frac{d^2x}{dt^2}$, $m\frac{d^2y}{dt^2}$, and $m\frac{d^2z}{dt^2}$.

The components of the velocity which tend to turn the body about the axis $Ox$ are $\frac{dy}{dt}$, with an arm $z$, and $\frac{dz}{dt}$, with an arm $y$ acting in the opposite direction; hence the angular momentum

$$\text{about } x \text{ is } m\left(z\frac{dy}{dt} - y\frac{dz}{dt}\right) = v_x,$$

$$\text{about } y \text{ is } m\left(x\frac{dz}{dt} - z\frac{dx}{dt}\right) = v_y,$$

$$\text{about } z \text{ is } m\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right) = v_z.$$

Their rates of variation in a unit of time are

$$\frac{dv_x}{dt} = m\left(z\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{dy}{dt} - y\frac{d^2z}{dt^2} - \frac{dy}{dt}\frac{dz}{dt}\right) = m\left(z\frac{d^2y}{dt^2} - y\frac{d^2z}{dt^2}\right),$$

$$\frac{dv_y}{dt} = m\left(x\frac{d^2z}{dt^2} = \frac{dx}{dt}\frac{dz}{dt} - z\frac{d^2x}{dt^2} - \frac{dz}{dt}\frac{dx}{dt}\right) = m\left(x\frac{d^2z}{dt^2} - z\frac{d^2x}{dt^2}\right),$$

$$\frac{dv_z}{dt} = m\left(y\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{dx}{dt} - x\frac{d^2y}{dt^2} - \frac{dx}{dt}\frac{dy}{dt}\right) = m\left(y\frac{d^2x}{dt^2} - x\frac{d^2y}{dt^2}\right).$$

Hence follow the equations which Rankine develops.

## PRINCIPLE OF D'ALEMBERT.

If we imagine any train of Mechanism in any way connected to which a set of external forces is applied, these external forces produce certain motions on being combined with the internal forces, which are brought into play by the motions of the parts, these motions are not the direct results of the external applied forces, but if a set of forces which would counteract these motions are applied to the machine, they stop the motion, and hence would counteract the effect of the external forces, or be in equilibrium with them. The forces which would directly produce these same motions are called the effective forces. Now it is evident that the internal force between any two of the parts is equal and opposite to the difference between the applied external force and the effective force at that point.

We are thus enabled to determine the internal forces acting between the parts of the system.

The components of one of the internal forces will therefore be

$$m\frac{d^2x}{dt^2} - F_x\,;\quad m\frac{d^2y}{dt^2} - F_y\,;\quad m\frac{d^2z}{dt^2} - F_z.$$

Rankine, Art. 568.

Rankine, Art. 569, should now be read carefully.

## ROTATIONS OF RIGID BODIES.

Suppose a rigid body (Fig. 29) to revolve about an axis perpendicular to the plane of the paper, and passing through O, imagine a particle of weight W situated at a perpendicular distance, $OA = r$, from the axis of rotation, and let the angular velocity of the body be $a$, then the particle at A will, in a unit of time, de-

scribe an arc $AB = ar$; hence its momentum, which is a measure of the force that has imparted to it this velocity, is $\frac{W}{g}ar$, and the moment of this force, or, in other words, its angular momentum with respect to the axis of rotation, is $\left(\frac{W}{g}ar\right)r = \frac{W}{g}ar^2 = a\frac{Wr^2}{g}$. The total angular momentum of the body is, consequently,

$$\Sigma a\frac{Wr^2}{g} = \frac{a}{g}\Sigma Wr^2;$$

the quantity $\Sigma Wr^2$ is called the moment of inertia of the body with reference to this axis of rotation, and when multiplied by $\frac{a}{g}$ gives us its angular momentum with reference to the same axis.

This illustration will suffice, I think, to make the student understand the application of the Moment of Inertia of a body with reference to an axis.

## RADIUS OF GYRATION.

Another Definition. The radius of gyration of a body with respect to an axis, is the perpendicular distance from the axis of that point at which, if its whole mass were concentrated, the angular momentum, and hence the moment of inertia, would remain unaltered.

If $\rho =$ the Radius of Gyration, the Moment of Inertia would be $\rho^2\Sigma W$, if the mass were all concentrated at this point; hence we must have,

$$\rho^2\Sigma W = \Sigma Wr^2.$$

$$\therefore \rho^2 = \frac{\Sigma Wr^2}{\Sigma W},$$

which gives the same result as Art. 574 of Rankine.

## EXAMPLES OF MOMENTS OF INERTIA.

The results are on page 518 of Rankine.

1. Find the moment of inertia of a sphere about its diameter.

Consider the sphere as made up of a series of thin plates, all horizontal, *i. e.*, parallel to the plane $xy$ (Fig. 30). Let one of

these plates be situated at a distance OA above the plane $xy$, then will AB be its radius.

From what we have learned previously, we know that the moment of inertia of the circle (radius AB) is,

$$\frac{\pi AB^4}{2} = \frac{\pi(OB^2 - OA^2)^2}{2} = \frac{\pi(a^2 - z^2)^2}{2},$$

$$\therefore\ I = \Sigma wr^2 = w\Sigma\frac{\pi}{2}(a^2 - z^2)^2dz = w\frac{\pi}{2}\int_{-a}^{a}(a^2 - z^2)^2dz$$

$$= \frac{\pi}{2}w\left\{a^4z - \frac{2a^2z^3}{3} + \frac{z^5}{5}\right\}_{-a}^{a} = \frac{8w\pi a^5}{15}\ \therefore \rho^2 = \frac{I}{W} = \frac{2a^2}{5}.$$

2. Spheroid of revolution. Let ABFG (Fig. 31) be the ellipse, which by its revolution around OZ generates the spheroid; let OB $= a$, OA $= r$. Equation of ellipse will be

$$\frac{x^2}{r^2} + \frac{z^2}{a^2} = 1\ \therefore\ x^2 = \frac{r^2}{a^2}(a^2 - z^2).$$

Imagine, as in the previous example, the spheroid to be formed of a series of circular horizontal plates, one of which is situated at a height OC $= z$, above the horizontal plane; the moment of inertia of the circle described by the revolution of CD will be

$$w\frac{\pi}{2}(CD)^4 = w\frac{\pi}{2}x^4 = w\frac{\pi r^4}{2a^4}(a^2 - z^2)^2;$$

hence the Moment of Inertia of the whole spheroid is

$$I = \Sigma wr^2 = \int_{-a}^{a} \omega\frac{\pi r^4}{2a^4}(a^2 - z^2)^2dz = \omega\frac{\pi r^4}{2a^4}\int_{-a}^{a}(a^2 - z^2)^2dz$$

$$= \omega\frac{\pi r^4}{2a^4}\cdot\tfrac{16}{15}a^5 = \frac{8\omega\pi ar^4}{15}.$$

$$\text{Area of circle} = \pi CD^2 = \pi x^2 = \frac{\pi r^2}{a^2}(a^2 - z^2)$$

$$\therefore\ \text{Volume of spheroid} = \int_{-a}^{a} w\frac{\pi r^2}{a^2}(a^2 - z^2)dz =$$

$$w\frac{\pi r^2}{a^2}\int_{-a}^{a}(a^2 - z^2)dz = \frac{w\pi r^2}{a^2}\left\{a^2z - \frac{z^3}{3}\right\}_{-a}^{a} = \tfrac{4}{3}\pi war^2$$

$$\therefore\ \rho^2 = \frac{I}{W} = \frac{2r^2}{5}.$$

3. Ellipsoid. Equation of Ellipsoid is,

$$\frac{x^2}{b^2} + \frac{y^2}{c^2} + \frac{z^2}{a^2} = 1.$$

Imagine it divided as before, into horizontal plates of elliptical shape, one of which is at a distance OA from O (Fig. 32).

Equation of ellipse in plane XZ will be

$$\frac{x^2}{b^2} + \frac{z^2}{a^2} = 1 \therefore \text{AB}^2 = x^2 = \frac{b^2}{a^2}(a^2 - z^2) \therefore \text{AB} = \frac{b}{a}\sqrt{(a^2 - z^2)}.$$

Equation of ellipse in plane YZ will be

$$\frac{y^2}{c^2} + \frac{z^2}{a^2} = 1 \therefore \text{AC}^2 = y^2 = \frac{c^2}{a^2}(a^2 - z^2) \therefore \text{AC} = \frac{c}{a}\sqrt{(a^2 - z^2)}.$$

Hence (Rankine, Art. 95) Moment of Inertia of Ellipse BCD about axis OZ is

$$\frac{\pi}{4}\cdot(\text{AB}^3\cdot\text{AC} + \text{AB}\cdot\text{AC}^3) = \frac{\pi\text{AB}\cdot\text{AC}}{4}(\text{AB}^2 + \text{AC}^2) =$$

$$\frac{\pi bc}{4a^2}(a^2 - z^2)\left\{\frac{b^2}{a^2}(a^2 - z^2) + \frac{c^2}{a^2}(a^2 - z^2)\right\} =$$

$$\frac{\pi bc}{4a^2}(a^2 - z^2)^2\left(\frac{b^2 + c^2}{a^2}\right) \therefore \text{I} = w\frac{\pi bc(b^2 + c^2)}{4a^4}\int_{-a}^{a}(a^2 - z^2)^2 dz =$$

$$w\frac{\pi bc(b^2 + c^2)}{4a^4}\cdot\frac{16a^5}{15} = \frac{4\pi wabc}{15}(b^2 + c^2).$$

$$\text{W} = w\int_{-a}^{a}\pi\text{AB}\cdot\text{AC}\cdot dz = \pi w\int_{-a}^{a}\frac{bc}{a^2}(a^2 - z^2)dz = \frac{\pi wbc}{a^2}\cdot\frac{4a^3}{3}$$

$$= \frac{4\pi wabc}{3} \therefore \rho^2 = \frac{\text{I}}{\text{W}} = \frac{b^2 + c^2}{5}.$$

4. Spherical shell. W and I will both be obtained by subtracting their values for the internal from those for the external sphere.

$$\therefore \text{I} = \frac{8\pi w(r^5 - r'^5)}{15}, \ \text{W} = \frac{4\pi w(r^3 - r'^3)}{3}, \ \rho^2 = \frac{\text{I}}{\text{W}} = \frac{2(r^5 - r'^5)}{5(r^3 - r'^3)}.$$

5. $r^3 - r'^3 = (r - r')(r^2 + rr' + r'^2) \qquad = dr(3r^2)$ nearly.

$r^5 - r'^5 = (r - r')(r^4 + r^3r' + r^2r'^2 + rr'^3 + r'^4) = dr(5r^4)$ nearly.

These results are obtained by making $r = r'$.

Now substitute in 4, and we have,

$$\text{I} = \frac{3\pi wr^4 dr}{3}, \ \text{W} = 4\pi wr^2 dr, \ \rho^2 = \frac{2r^2}{3}.$$

6. Circular cylinder. Let (Fig. 33) $OA = a$, $OB = r$. Let, as before, the cylinder be divided into a set of thin horizontal circular plates, one of which is situated at a height $OC = z$, above O. Moment of inertia of this plate is $w\frac{\pi r^4}{2}dz$.

$$\therefore I = \int_{-a}^{a} \frac{w\pi r^4}{2}dz = \pi w r^4 a.$$

$$W = w\int_{-a}^{a} \pi r^2 dz = 2\pi w a r^2 \ \therefore \ \rho^2 = \frac{r^2}{2}.$$

7. In the Elliptic cylinder the moment of inertia of one of the plates is $w\frac{\pi bc}{4}(b^2 + c^2)dz$.

$$\therefore I = \frac{\pi w bc(b^2 + c^2)}{4}\int_{-a}^{a} dz = \frac{\pi wabc}{2}(b^2 + c^2).$$

$$W = 2w\pi bca \ \therefore \ \rho^2 = \frac{b^2 + c^2}{4}.$$

8. Found by subtraction.

9. Found from 8, like 5.

10. Circular Cylinder (Fig. 34) about the axis OZ; let $OD = r$, $OE = a$. Our horizontal sections will now be rectangles, and the moment of inertia will be $\frac{1}{3}w AC \cdot AB(AB^2 + AC^2)$. Now

$$AC^2 = OC^2 - OA^2 = r^2 - z^2$$

$\therefore$ Moment of inertia of plate is $\frac{4}{3}wa\sqrt{(r^2 - z^2)}\,(a^2 + r^2 - z^2)$, hence

$$I = \int_{-r}^{r} \tfrac{4}{3}wa\sqrt{(r^2 - z^2)}\{(a^2 + r^2) - z^2\}dz$$

$$= \tfrac{4}{3}wa(a^2 + r^2)\int_{-r}^{r}\sqrt{(r^2 - z^2)}dz - \tfrac{4}{3}wa\int_{-r}^{r} z^2\sqrt{(r^2 - z^2)}dz$$

$$= \tfrac{4}{3}wa\left\{\tfrac{1}{4}(r^2 - z^2)^{\frac{3}{2}} + (a^2 + r^2)\int\sqrt{(r^2 - z^2)}dz - \frac{r^2}{4}\int\sqrt{(r^2 - z^2)}dz\right\}_{-r}^{r}$$

$$= \tfrac{4}{3}wa\left\{\left(a^2 + \frac{3r^2}{4}\right)\frac{\pi r^2}{2}\right\} = \frac{\pi w a r^2(3r^2 + 4a^2)}{6}.$$

$$W = 2\pi w a r^2 \ \therefore \ \rho^2 = \frac{I}{W} = \frac{r^2}{4} + \frac{a^2}{3}.$$

11. If the circle becomes an ellipse, we shall have,

$$AC = y, \text{ and } \frac{y^2}{c^2} + \frac{z^2}{b^2} = 1 \therefore AC = \frac{c}{b}\sqrt{(b^2 - z^2)}.$$

$$\therefore I = \int_{-b}^{b} \tfrac{4}{3} wa \frac{c}{b}\sqrt{(b^2 - z^2)}\left(a^2 + c^2 - \frac{c^2 z^2}{b^2}\right) dz =$$

$$\tfrac{4}{3}\frac{wac}{b}\left\{(a^2 + c^2)\int_{-b}^{b}\sqrt{(b^2 - z^2)}dz - \frac{c^2}{b^2}\int_{-b}^{b} z^2\sqrt{(b^2 - z^2)}dz\right\}$$

$$= \tfrac{4}{3}\frac{wac}{b}\left\{\frac{c^2}{4b^2}(b^2 - z^2)^{\frac{3}{2}} + \left(a^2 + c^2 - \frac{c^2}{4}\right)\int_{-b}^{b}\sqrt{(b^2 - z^2)}dz\right\}$$

$$= \tfrac{4}{3}\frac{wac}{6}\left(\frac{4a^2 + 3c^2}{4}\right)\frac{\pi b^2}{2} = \frac{\pi wab^2(3c^2 + 4a^2)}{6}.$$

12. Found by subtraction.

13. Deduced from 12, like 5.

14. $$I = \int_{-a}^{a} \tfrac{4}{3} wbc(b^2 + c^2)dz = \frac{8wabc}{3}(b^2 + c^2).$$

15. We must first find the moment of inertia of a rhombus. To do this we have (Fig. 35),

$$\frac{y}{c - x} = \frac{b}{c} \therefore y = \frac{b}{c}(c - x)$$

$$\therefore I = \int\int x^2 dx dy = \frac{2b}{c}\int_{-c}^{c} x^2(c - x)dx = \frac{4bc}{3}(b^2 + c^2),$$

hence, for the rhombic prism,

$$I = \int_{-a}^{a}\frac{4wbc}{3}(b^2 + c^2)dz = \frac{8wabc(b^2 + c^2)}{3}.$$

$$W = 8wabc \therefore \rho^2 = \frac{b^2 + c^2}{3}.$$

16. OD $= b$ (Fig. 36), AC $= a$, OE $= c$, $\therefore$ BA $= y$.

$$\frac{y}{b - z} = \frac{c}{b} \therefore y = \frac{c}{b}(b - z),$$

$\therefore$ Moment of inertia of rectangle is

$$\tfrac{4}{3}AB \cdot AC(AB^2 + AC^2) = \tfrac{4}{3}\frac{ac}{b}(b - z)\left\{\frac{c^2}{b^2}(b - z)^2 + a^2\right\}$$

$$= \frac{4ac^3}{3b^3}(b - z)^3 + \frac{4a^3c}{3b}(b - z)$$

$$\therefore I = 2w\int_0^b \frac{4ac^3}{3b^3}(b - z)^3 dz + 2\frac{4a^3c}{3b}w\int_0^b (b - z)dz$$

$$= \frac{2wabc(c^2 + 2a^2)}{3}.$$

$$W = 4wabc \therefore \rho^2 = \frac{I}{W} = \frac{c^2}{6} + \frac{a^2}{3}.$$

Arts. 575 and 576 of Rankine are perfectly clear, and should be compared with Art. 95.

Note on Art. 576, Cor. I. From Equation 2 of Art. 576,

$$I = r_0^2 \Sigma W + I_0, \text{ but } I = \rho^2 \Sigma W;\ I_0 = \rho_0^2 \Sigma W$$

$$\therefore \rho^2 \Sigma W = r_0^2 \Sigma W + \rho_0^2 \Sigma W \qquad \therefore \rho^2 = r_0^2 + \rho_0^2,$$

which shows that $\rho$ is the hypothenuse of a right triangle, of which $r_0$ and $\rho_0$ are the sides.

Cor. II. Equation 2 gives $I = r_0^2 \Sigma W + I_0$,

$$\therefore I_0 = I - r_0^2 \Sigma W.$$

Cor. III. Equation 2 itself shows this.

Art. 577 should be read; and the examples of Art. 578 have already been performed. Art. 579 is clear, if carefully studied, only there are, first, one letter, C, left off in the figure, and an exponent on the $c$ in $\frac{b^2 + c^2}{6}$.

EXAMPLE.

Find the radius of gyration of a quadrant sector of a circular cylinder about an axis passing through its centre of gravity.

## CENTRE OF PERCUSSION.

The explanation of *Centre of Percussion* given by Rankine, should be carefully studied, but I shall present it in a different aspect, which I think will furnish a clearer conception of it.

First, imagine a body (Fig. 37) revolving about an axis perpendicular to the plane of the paper, and passing through O with an angular velocity $a$. If, with O as a centre, and a radius $OA = r$, we describe an arc BC, all particles situated on this arc have a linear velocity, $ar$, which must have been generated by a force which may be measured by $war$, where $w$ is the weight of the particle; the moment of the force is $war^2$, and the total angular momentum of the body is $\Sigma war^2 = a\Sigma wr^2 = aI$. The sum of the forces acting on the body is $\Sigma war = a\Sigma wr$, hence the point of appli-

cation of the resultant single force which would produce this rotation is situated at H, when $AH = \frac{a\Sigma wr^2}{a\Sigma wr} = \frac{I}{\Sigma wr}$, the magnitude of that force being $a\Sigma wr$.

This point of application of the resultant single force required to produce the angular velocity $a$ is the centre of percussion.

## ANOTHER VIEW.

### INSTANTANEOUS AXIS.

Imagine a body (Fig. 38), and for the sake of simplicity let the particles be conceived to be distributed along a single line AB, and suppose a force F applied at D. We may conceive two equal and opposite forces, each equal to F, applied at C, the centre of gravity of AB, without altering the motion of the body. Then these three forces are equivalent to a single force equal to F, applied at the centre of gravity C, which produces translation of the whole body, and, secondly, a couple whose moment is F·CD, whose effect is to produce rotation around an axis passing through the centre of gravity C. Under this condition of things it is evident that the centre of gravity C will move forward with a velocity due to the force F, the point D will move farther forward, while those on the other side of C will have a less and less velocity, and the particle situated at A (if the force is not too great) will move backward, hence there must be some point, which, for the instant in question, is at rest (*i.e.*, which is changing from moving in one direction to move in another), or about which, for the moment, the body is rotating, and if this point were fixed there would be no strain on the pivot, caused by the force applied at D.

An axis passing through this point is called the *Instantaneous Axis*, or the axis about which the body at that moment tends to rotate. This point is evidently the point of application of the resultant of the forces, which give the body its angular momentum about the axis, and hence the point D is the *Centre of Percussion* of the body in reference to E.

Fig. 39 will make the condition of things clear. Let AB represent the body, C its centre of gravity, D the point of application of the force F, and E the instantaneous axis (D is then the centre of percussion in reference to an axis through E). Let CG represent the distance through which the centre of gravity moves in a short time, $dt$, impelled by the force F, then $CG = \frac{F}{\Sigma w}dt$.

Then the point D will, in virtue of the translation, in common with the centre of gravity, move to H, and in virtue of its rotation around this centre of gravity, describe the arc HK, such that $a\Sigma wr^2 = F{\cdot}CD \therefore a = \frac{F{\cdot}CD}{\Sigma wr^2}$, ($a$ being = the angular velocity); then the arc $HK = aGH{\cdot}dt = \frac{F{\cdot}CD}{\Sigma wr^2} GH{\cdot}dt = \frac{F{\cdot}CD^2}{\Sigma wr^2}dt$; but for a short time $HL = HK$, nearly, hence

$$EC : GH = CG : HL,$$

$$\text{or } EC : CD = \frac{F}{\Sigma w}dt : \frac{F{\cdot}CD^2}{\Sigma wr^2}dt \;\therefore\; \frac{EC}{DC} = \frac{\Sigma wr^2}{CD^2{\cdot}\Sigma w} \therefore$$

$$EC{\cdot}CD = \frac{\Sigma wr^2}{\Sigma w} \therefore CD = \frac{\Sigma wr^2}{EC{\cdot}\Sigma w}$$

$$\therefore DE = CD + EC = EC + \frac{\Sigma wr^2}{EC{\cdot}\Sigma w} = \frac{EC^2\Sigma w + \Sigma wr^2}{EC{\cdot}\Sigma w} = \frac{I}{\Sigma wr},$$

the same expression obtained before.

From the value already obtained for the distance from the axis of rotation to the centre of percussion, viz., $CB = \frac{\Sigma wr^2}{\Sigma wr}$, Rankine, Fig. 239), we deduce

$$CB = \frac{r_0^2\Sigma w + I_0}{r_0\Sigma w} = \frac{I}{r_0\Sigma w} = \frac{1}{r_0}\frac{\Sigma wr^2}{\Sigma w} = \frac{\rho^2}{r_0} \therefore \rho^2 = r_0 CB,$$

*i.e.*, the radius of gyration is a mean proportional between the distance EC and the distance $r_0$, between the axis of rotation and the centre of gravity. Hence follows the construction given in Rankine (Fig. 239, p. 520), viz., at G draw GD perpendicular to CB, and equal to $\rho_0$, then join CD, then $CD^2 = \rho_0^2 + r_0^2 = \rho^2$, then at D draw CB perpendicular to CD, and B is the centre of percussion. for $CD^2 = CG{\cdot}CB$, or $\rho^2 = r_0 CB \therefore CB = \frac{\rho^2}{r_0}$.

If the axis of rotation passes through the centre of gravity of the body, $r_0 = 0$ and $CB = \infty$, hence the centre of percussion moves off to an infinite distance, or in other words there is no centre of percussion (Rankine, Art. 582).

EXPLANATION OF THE WORK OF ART. 583, RANKINE.

Let $x, y, z$ (Fig. 40), be the three original axes, and $x', y', z'$, the new axes, and let P be the point in consideration, whose coördinates with reference to the old axes are $OB = x$, $BA = y$, $AP = z$, and $OC = x'$, is one of its coördinates with reference to the new axes. Now if CP be projected on $Ox'$, the length of the projection will be $OC = x'$.

If instead of that we project the broken line, OBAP, we shall have,

$$OB \cos xOx' + AB \cos yOx' + AP \cos zOx'$$
$$= x \cos xx' + y \cos yx' + z \cos zx', \text{ hence}$$
$$x' = x \cos xx' + y \cos yx' + z \cos zx', \text{ so also}$$
$$y' = x \cos xy' + y \cos yy' + z \cos zy', \text{ and}$$
$$z' = x \cos xz' + y \cos yz' + z \cos zz'.$$

In the same way we should obtain

$$x = x' \cos x'x + y' \cos y'x + z' \cos z'x.$$
$$y = x' \cos x'y + y' \cos y'y + z' \cos z'y.$$
$$z = x' \cos x'z + y' \cos y'z + z' \cos z'z.$$

The following equations have already been proved in the Statics.

$$\cos^2 xx' + \cos^2 xy' + \cos^2 xz' = 1.$$
$$\cos^2 yx' + \cos^2 yy' + \cos^2 yz' = 1.$$
$$\cos^2 zx' + \cos^2 zy' + \cos^2 zz' = 1.$$
$$\cos^2 xx' + \cos^2 yx' + \cos^2 zx' = 1.$$
$$\cos^2 xy' + \cos^2 yy' + \cos^2 zy' = 1.$$
$$\cos^2 xz' + \cos^2 yz' + \cos^2 zz' = 1.$$

The proof of the following equations is precisely similar to that on page 31 and 32 of the 1st part on Statics.

$$\cos yx' \cos zx' + \cos yy' \cos zy' + \cos yz' \cos zz' = 0.$$
$$\cos xx' \cos zx' + \cos xy' \cos zy' + \cos xz' \cos zz' = 0.$$
$$\cos xx' \cos yx' + \cos xy' \cos yy' + \cos xz' \cos yz' = 0.$$

Also the following:

$$\cos xy' \cos xz' + \cos yy' \cos yz' + \cos zy' \cos zz' = 0.$$
$$\cos xz' \cos xx' + \cos yz' \cos yx' + \cos zz' \cos zx' = 0.$$
$$\cos xx' \cos xy' + \cos yx' \cos yy' + \cos zx' \cos zy' = 0.$$

Let us have, for the sake of brevity,

$$\cos xx' = l_1, \quad \cos yx' = l_2, \quad \cos zx' = l_3.$$
$$\cos xy' = m_1, \quad \cos yy' = m_2, \quad \cos zy' = m_3.$$
$$\cos xz' = n_1, \quad \cos yz' = n_2, \quad \cos zz' = n_3.$$

And the last three equations become,

$$m_1n_1 + m_2n_2 + m_3n_3 = 0. \quad (1.)$$
$$n_1l_1 + n_2l_2 + n_3l_3 = 0. \quad (2.)$$
$$l_1m_1 + l_2m_2 + l_3m_3 = 0. \quad (3.)$$

Also $\quad l_1^2 + l_2^2 + l_3^2 = 1. \quad (4.)$

Eliminating $l_2$ from (2.) and (3.), we have,

$$\frac{l_1}{l_3} = \frac{m_2n_3 - m_3n_2}{m_1n_2 - m_2n_1};$$

and eliminating $l_1$ we have,

$$\frac{l_2}{l_3} = \frac{m_3n_1 - m_1n_3}{m_1n_2 - m_2n_1}.$$

From (4.) we have,

$$\frac{l_1^2}{l_3^2} + \frac{l_2^2}{l_3^2} + 1 = \frac{1}{l_3^2} \quad \therefore \quad \frac{(m_2n_3 - m_3n_2)^2 + (m_3n_1 - m_1n_3)^2}{(m_1n_2 - m_2n_1)^2} + 1 = \frac{1}{l_3^2}$$

$$\therefore l_3^2 = \frac{(m_1n_2 - m_2n_1)^2}{(m_2n_3 - m_3n_2)^2 + (m_3n_1 - m_1n_3)^2 + (m_1n_2 - m_2n_1)^2};$$

but the denominator is equal to

$$m_1^2(n_2^2 + n_3^2) + m_2^2(n_1^2 + n_3^2) + m_3^2(n_1^2 + n_2^2) - 2m_1m_2n_1n_2$$
$$- 2m_1m_3n_1n_3 - 2m_2m_3n_2n_3$$
$$= m_1^2(1 - n_1^2) + m_2^2(1 - n_2^2) + m_3^2(1 - n_3^2)$$
$$- 2(m_1n_1m_3n_3 + m_1n_1m_2n_2 + m_2n_2m_3n_3).$$

Now squaring (1.) we have

$$(m_1n_1 + m_2n_2 + m_3n_3)^2 = m_1^2n_1^2 + m_2^2n_2^2 + m_3^2n_3^2$$
$$+ 2(m_1n_1m_2n_2 + m_1n_1m_3n_3 + m_2n_2m_3n_3) = 0.$$

$$\therefore -2(m_1n_1m_2n_2 + m_1n_1m_3n_3 + m_2n_2m_3n_3) = m_1^2n_1^2 + m_2^2n_2^2 + m_3^2n_3^2.$$

$\therefore$ The above denominator reduces to

$$1 - (m_1^2n_1^2 + m_2^2n_2^2 + m_3^2n_3^2) + m_1^2n_1^2 + m_2^2n_2^2 + m_3^2n_3^2 = 1$$

$$\therefore l_3 = m_1n_2 - m_2n_1, \text{ and}$$
$$l_2 = m_3n_1 - m_1n_3,$$
$$l_1 = m_2n_3 - m_3n_2. \qquad \text{Q. E. D.}$$

These formulæ enable us to obtain the moments of Inertia and moments of Deviation, with reference to any new set of rectangular axes.

Note on Art. 584 of Rankine.

The derivation of Equation (4.) from Equation (3.) is precisely analogous to that of Equation (4.) from (3.), Art. 107, Rankine.

Every cubic equation has at least one real root, since imaginary roots always enter an equation in pairs, and hence there is always one value of $Sx_1^2$, and one position of the axes, for which the moments of deviation are 0. Now by interchanging the axes of $x$, $y$ and $z$, we shall find that $Sx_1^2$, $Sy_1^2$, $Sz_1^2$, will be interchanged, and hence these must be the 3 roots of the cubic equation; hence all the roots are real, and since there are 3 variations in sign, they are positive.

To obtain equations (6.) multiply the 1st of (3.) by $Syz$, and the 2d by $Sxz$, and we obtain

$$\cos xx_1\{(Sx^2 - Sx_1^2)Syz - Sxy \cdot Szx\}$$
$$+ \cos yy_1\{Sxy\,Syz - (Sy^2 - Sx_1^2)Szx\} = 0.$$
$$\therefore \frac{\cos xx_1}{\cos yy_1} = \frac{(Sx_1^2 - Sy^2)Szx + Sxy\,Syz}{(Sx_1^2 - Sx^2)Syz + Sxy\,Szx}, \text{ or}$$
$$\cos xx_1 : \cos yy_1 = \frac{1}{(Sx_1^2 - Sx^2)Syz + Sxy\,Szx} : \frac{1}{(Sx_1^2 - Sy^2)Szx + Sxy\,Syz},$$

and, in like manner, eliminating $\cos yy_1$ we obtain,

$$\cos xx_1 : \cos zz_1 = \frac{1}{(Sx_1^2 - Sx^2)Syz + Sxy\,Szx} : \frac{1}{(Sx_1^2 - Sz^2)Sxy + Syz\,Szx}.$$

The sum of the roots $= A = Sx_1^2 + Sy_1^2 + Sz^2 = SR^2$, and so the sum of the products of the roots taken two and two is,

$$B = Sy_1^2Sz_1^2 + Sz_1^2Sx_1^2 + Sx_1^2Sy_1^2, \text{ and } C = Sx_1^2Sy_1^2Sz_1^2.$$

Note on Art. 585.

To prove that $Sx^2 = (\cos^2 \alpha)Sx_1^2 + (\cos^2 \beta)Sy_1^2 + (\cos^2 \gamma)Sz_1^2$, we have, $x = x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma$; consequently,

$$x^2 = x_1^2 \cos^2 \alpha + y_1^2 \cos^2 \beta + z_1^2 \cos^2 \gamma + 2x_1y_1 \cos \alpha \cos \beta +$$
$$2x_1z_1 \cos \alpha \cos \gamma + 2y_1z_1 \cos \beta \cos \gamma.$$
$$\therefore \int\!\!\int\!\!\int x^2 dxdydz = \cos^2 \alpha \int\!\!\int\!\!\int x_1^2 dx_1dy_1dz_1 + \cos^2 \beta \int\!\!\int\!\!\int y_1^2 dx_1dy_1dz_1$$
$$+ \cos^2 \gamma \int\!\!\int\!\!\int z_1^2 dx_1dy_1dz_1 + 2 \cos \alpha \cos \beta \int\!\!\int\!\!\int x_1y_1dx_1dy_1dz_1$$
$$+ 2 \cos \alpha \cos \gamma \int\!\!\int\!\!\int x_1z_1dx_1dy_1dz_1 + 2 \cos \beta \cos \gamma \int\!\!\int\!\!\int y_1z_1dx_1dy_1dz_1;$$

that is. $Sx_1^2 = Sx_1^2 \cos^2\alpha + Sy_1^2 \cos^2\beta + Sz_1^2 \cos^2\gamma +$
$2Sx_1y_1 \cos\alpha \cos\beta + 2Sx_1z_1 \cos\alpha \cos\gamma + 2Sy_1z_1 \cos\beta \cos\gamma$;
but since the body is referred to its principal axes,

$$Sx_1y_1 = 0,\ Sx_1z_1 = 0, \text{ and } Sy_1z_1 = 0.$$

$$\therefore Sx^2 = Sx_1^2 \cos^2\alpha + Sy_1^2 \cos^2\beta + Sz_1^2 \cos^2\gamma.$$

Equations (2.) have been obtained in Art. 583.

$$\therefore Sx^2 = SR^2 - I,\quad Sx_1^2 = SR^2 - I_1,\quad Sy_1^2 = SR^2 - I_2,$$

$$Sz_1^2 = SR^2 - I_3 \ldots SR^2 - I = SR^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$

$$- I_1 \cos^2\alpha - I_2 \cos^2\beta - I_3 \cos^2\gamma$$

$$\therefore\ I = I_1 \cos^2\alpha + I_2 \cos^2\beta + I_3 \cos^2\gamma.$$

The remainder of Art. 585 needs no comment.

Note on Art. 586, Rankine.

Equation (2.) is derived in a manner similar to Equation (1.) of the last article.

The equation of an ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the equation of a plane tangent to it at the point $x_1 y_1 z_1$ is,

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1,$$

and the length of the perpendicular from the origin on the plane is

$$n = \frac{1}{\sqrt{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4}}},\ \text{but}$$

$$x_1 = s\cos\alpha,\ y_1 = s\cos\beta,\ z_1 = s\cos\gamma\ \therefore$$

$$\frac{1}{n^2} = s^2\left(\frac{\cos^2\alpha}{a^4} + \frac{\cos^2\beta}{b^4} + \frac{\cos^2\gamma}{c^4}\right)$$

$$\therefore\ \frac{\cos^2\alpha}{a^4} + \frac{\cos^2\beta}{b^4} + \frac{\cos^2\gamma}{c^4} = \frac{1}{s^2n^2}. \qquad \text{Q. E. D.}$$

## UNIFORM ROTATION.

(Art. 588, Rankine.) To illustrate the conclusion of this article, we must remember that couples are compounded by the parallelogram of forces by means of their moment axes, hence a tendency

to rotate about two different axes, results (if the body is free) in a rotation about neither of the two, but about one intermediate in position between the two.

Now suppose we have a wire, AB (Fig. 41), compelled to rotate around the axis $Oz$, the forces necessary to act on the wire to cause this rotation, are a set of forces at all points of the wire (perpendicular to the plane of the paper), and proportional to their distances from $Oz$; but these forces have also a moment relatively to the axis $Ox$, equal to the resultant force multiplied by its distance from $Ox$; hence it has also a tendency to rotate around $Ox$, and if the body were free it would rotate around neither $Oz$ nor $Ox$, but around an axis intermediate between the two, and its axis of rotation would continue shifting till the wire came to rotate around a line* perpendicular to AB: this is an axis of inertia, and that where the moment of inertia is greatest. Thus, if the axis of rotation, or the axis about which the body is rotating, is not an axis of inertia, the body has a tendency to rotate around other axes also, at right-angles to this (and hence its axis of rotation changes position if the body is free, in which case the axis of rotation is merely an instantaneous axis).

In the case of a free body the axis of rotation, or instantaneous axis, moves, and comes to change its position so as to approach an axis of inertia, and it only becomes stable when it has reached that axis. If the body is not free the tendency to rotate about other axes causes a stress on the axis which has to be resisted by the strength of the material, or else it will bend, and ultimately break the axis. Hence, in machinery, other things being equal, it is better that all the axes of the wheels, and all the bearings of the rotating pieces should be axes of inertia of the rotating pieces, as this avoids the bending stress on these bearings.

To illustrate farther the composition of rotations, we may consider the motion of the gyroscope.

Suppose the gyroscope (Fig. 42) to have impressed upon it a rotation in the direction of the arrow, it will have, also, another tendency to rotate, caused by the force of gravity; the first rota-

tion about $Ox$, and the second about $Oy$; these two rotations compounded by the parallelogram give a rotation about an intermediate axis OP, in the plane $xy$, hence the whole arm has a retrograde motion in a horizontal plane, or about an axis OZ, in addition to its tendency to rotate about an axis parallel to OP: and the result of compounding these two would be to cause rotation about an inclined axis, or, in other words, the axis of rotation has risen.

Fig. 43 will serve as a figure for Rankine's work in Art. 588. With the latter part of this article the student should compare Art. 95 of Rankine's statics, as they are precisely analogous.

Note on Art. 589, Rankine.

We have $E = \frac{v^2W}{2g}$, and also $E = \frac{a^2I}{2g}$; hence the actual energy bears the same relation to linear velocity and weight that it does to angular velocity and moment of Inertia.

Again, from (8.) we see that $I \propto \frac{1}{OR^2}$, hence

$$E = \frac{a^2I}{2g} \propto \frac{a^2}{2g \cdot OR}, \text{ which is (3).}$$

Note on Art. 590.

I. The result of applying any set of forces to a rigid body is to impart to the body (in the most complex case) a translation and a rotation combined, the rotation being in a plane perpendicular to the direction of the translation; leaving out of account the translation which affects the position of the centre of gravity of the body, we have a rotation around a certain line; hence, considering the centre of gravity as a fixed point, the motion is confined to one plane, and a perpendicular to this plane is the axis of angular momentum.

II. If no force acts on the body its angular momentum can not change.

III. Refer to Art. 565 of Bankine.

Equations (1.) are respectively (4.) of Art. 588, and (2.) of Art. 589, Rankine.

Now to derive (2.) we have,

$$a^2 = \frac{A^2 g^2}{I^2 + K^2}, \text{ and } a^2 = \frac{2gE}{I} \quad \therefore \quad \frac{A^2 g^2}{I^2 + K^2} = \frac{2gE}{I}$$

$$\therefore \frac{A^2 g}{2E} = \frac{I^2 + K^2}{I}, \text{ and from (2.) of Art. 588,}$$

$$I^2 + K^2 = I_1^2 \cos^2 \alpha + I_2^2 \cos^2 \beta + I_3^2 \cos^2 \gamma$$

$$\therefore \frac{gA^2}{2E} = \frac{I^2 + K^2}{I} = \frac{I_1^2 \cos^2 \alpha + I_2^2 \cos^2 \beta + I_3^2 \cos^2 \gamma}{I_1 \cos^2 \alpha + I_2 \cos^2 \beta + I_3 \cos^2 \gamma}$$

but A is constant, and also E $\therefore \frac{I^2 + K^2}{I}$ is constant.

The second of equations (1.) gives,

$$a^2 = \frac{2gE}{I} \quad \therefore \quad a = \frac{\sqrt{(2gE)}}{\sqrt{I}},$$

but the numerator is constant, and from equation (8.) of Art. 588, $I \propto \frac{1}{OR^2} \therefore \sqrt{I} \propto \frac{1}{OR}$, and hence $a \propto$ OR.

From (2.) and (3.) combined we may have,

$$\frac{I_1^2 \cos^2 \alpha + I_2^2 \cos^2 \beta + I_3^2 \cos^2 \gamma}{I_1 \cos^2 \alpha + I_2 \cos^2 \beta + I_3 \cos^2 \gamma} = \frac{1}{ON^2}$$

$$\therefore I_1^2 \cos^2 \alpha + I_2^2 \cos^2 \beta + I_3^2 \cos^2 \gamma =$$

$$\frac{I_1}{ON^2} \cos^2 \alpha + \frac{I_2}{ON^2} \cos^2 \beta + \frac{I_3}{ON^2} \cos^2 \gamma \therefore$$

$$\left(I_1^2 - \frac{I_1}{ON^2}\right)\cos^2 \alpha + \left(I_2^2 - \frac{I_2}{ON^2}\right)\cos^2 \beta + \left(I_3^2 - \frac{I_3}{ON^2}\right)\cos^2 \gamma = 0.$$

Note on Art. 591 and 592.

To illustrate the deviating couple explained in the above articles, let us imagine an ellipse (Fig. 44) constrained to revolve around an axis $Oy$ passing through its centre O, but which is neither the transverse nor the conjugate axis; the forces which act on the different particles to impart to them this rotation are perpendicular to the plane of the paper, and proportional in magnitude to the distances of these particles from $Oy$; thus the force acting on a particle C, at a distance EC $= x$ from $Oy$, will be $\frac{W}{g} ax$ (W being the weight of the particle, and $a$ the angular velocity of the

body), and the linear velocity of the particle will be $ax$; but (Art. 537. Rankine) the centrifugal force of this particle, or the force acting on it in the direction EC, is $\frac{Wv^2}{gr}$, equal in this case to

$$\frac{W}{g}\frac{a^2x^2}{x} = \frac{Wa^2x}{g}.$$

The moment of this force with reference to an axis passing through O, perpendicular to the plane of the paper, is

$$\frac{Wa^2xy}{g};$$

hence the total moment of the couple tending to turn the body about this latter axis, is

$$\Sigma\frac{Wa^2xy}{g} = \frac{a^2}{g}\Sigma Wxy = \frac{a^2K}{g},$$

which proves Rankine's Equation (1.) of Art. 592. The remainder of the article is very easily deduced.

Note on Art. 593.

Let Fig. 45 represent a couple, of force F, and arm $l$, and suppose that while it is acting the body moves through an angle $a$, then the amount of work done will be,

$$F{\cdot}AA' + F{\cdot}BB' = F\theta OA + F\theta OB = \theta F(OA + OB) = \theta Fl = Fl\theta.$$

Note on Art. 593, Rankine.

In the case in question, $\theta = di$, hence the energy exerted is $Fldi = Mdi$; but $di = adt$; heuce energy exerted $= Madt$.

When the axis of the couple applied to a body makes an angle $\varphi$ with the axis of rotation, the only component of the couple that acts in the direction of motion of the body is $M \cos \varphi$; hence the energy exerted is $Ma \cos \varphi dt$, as stated in expression (2.) of Rankine.

## VARIED ROTATION.

Art. 594, Rankiue.

The couple whose moment is $M \cos \psi$ acts in a plane perpendicular to the axis of angular momentum, and hence causes change

in the body's angular momentum. The couple M sin $\varphi$, on the other hand, tends to cause rotation about an axis perpendicular to the axis of angular momentum, and hence to change the position of the axis of angular momentum.

For the value $A = \frac{a}{g}\sqrt{(I^2 + K^2)}$ refer to Art. 588, Rankine; and for Equation (5.), Art. 594, refer to (4.), Art. 593, and (1.), Art. 590.

Note on Art. 597.

Equation (1.) may be deduced as follows: from Equation (1.), Art. 595, we have, when $\varphi = 0$,

$$M = \frac{Ida}{gdt} \therefore da = \frac{Mgdt}{I},$$

or by integrating, and observing that when $t = 0$, $a = a_0$, we have,

$$a = a_0 + \frac{Mgt}{I}; \text{ again, } \frac{di}{dt} = a \therefore \frac{di}{dt} = a_0 + \frac{Mgt}{I}; \text{ or}$$

$$di = a_0 dt + \frac{Mgtdt}{I}.$$

Integrating, and observing that when $t = 0$, $i = 0$, we have,

$$i = a_0 t + \frac{Mgt^2}{2I}.$$

$$\text{Again } a - a_0 = \frac{Mgt}{I}, \text{ and } \therefore a + a_0 = 2a_0 + \frac{Mgt}{I},$$

$$\therefore a^2 - a_0^2 = \frac{Mgt}{I}\left(2a_0 + \frac{Mgt}{I}\right) = \frac{2Mg}{I}\left(a_0 t + \frac{Mgt^2}{I}\right) = \frac{2Mgi}{I}$$

$$\therefore I(a^2 - a_0^2) = 2Mgi, \text{ or } Mi = \frac{I(a^2 - a_0^2)}{2g}.$$

## EXAMPLES.

1. Given a cast iron flywheel, whose external radius = 20 ft., internal radius = 18 ft., thickness = 2 ft., and of which the matter is supposed to be all concentrated in the rim. Suppose the above wheel to have a speed of 10 turns per minute, how much energy was exerted to give it this speed, and how much work is it capable of transmitting?

2. Suppose another similar wheel on the same shaft to be moving with the above wheel, and to be suddenly disengaged, what is the moment of the couple required to stop its rotation while it is making one turn? Given the dimensions of this latter wheel as follows: —

External radius = 10 ft., internal radius = 9 ft., thickness = 1 ft.; the whole mass being, as before, regarded as concentrated in the rim.

Note on Art. 598.

$i_1$ corresponds to $r$, for it is the extreme angular displacement.

$i$ corresponds to $x$, for it is the variable angular displacement.

$M_1$ corresponds to Q, for it is the moment of the couple acting.

$\frac{M_1}{i_1}$ corresponds to $\frac{Wa^2}{g}$, for $M = -\frac{M_1 i}{i_1}$, and $Q = -\frac{Wa^2 x}{g}$;

but $i$ corresponds to $x$ $\therefore$ $\frac{M_1}{i_1}$ corresponds to $\frac{Wa^2}{g}$.

M corresponds to Q, for it is the deviating couple.

$a$ corresponds to $\frac{dx}{dt}$, for it is the angular velocity.

$\frac{gM_1}{i_1 I}$ corresponds to $a^2$, for

$$\frac{M_1}{i_1} \text{ corresponds to } \frac{Wa^2}{g}, \text{ and I to W.}$$

$$\therefore \frac{gM_1}{i_1 I} \text{ corresponds to } a^2.$$

In connection with Arts. 601–605 of Rankine, I would refer the student to pages 45–47 of this volume.

Note on Art. 606.

Let the respective quantities be denoted by letters, as Rankine states them, and draw the vertical axis DG. Suppose that the body were rotating about the axis DG; there would be two sources of deviating force. 1st, a tendency of the axis EG to rotate till it should coincide with DG; 2d, a tendency to bring the axis GC into coincidence with DG.

($a$.) If the body were rotating about the axis EG. The resultant centrifugal force would be $\frac{Wa^2\rho_1}{g}$, and would act at a point F, where GF $= \rho_1$, in a direction perpendicular to EG. The component of this force in the direction MF, or $\frac{Wa^2\rho_1 \sin a}{g}$, is the deviating force caused by the first tendency, and its moment about G is $\frac{Wa^2\rho_1 \sin a}{g} \cdot \text{GM} = \frac{Wa^2}{g}\rho_1^2 \sin a \cos a$.

($b$.) So likewise the moment of the deviating force caused by the second tendency is, $\frac{Wa^2}{g}\rho_2^2 \sin a \cos a$, and acts in the opposite direction; hence the resultant moment of the deviating couple, when the body rotates about DG, is $\frac{Wa^2}{g}(\rho_1^2 - \rho_2^2) \sin a \cos a = \frac{Wa^2}{g}(\text{I}_1 - \text{I}_2) \cos a \sin a$.

When the body rotates about the axis CX, the moment of the deviating couple due to its weight is $\frac{Wa^2}{g} r_0^2 \cos a \sin a$, hence the total moment of the deviating couple is

$$\frac{Wa^2}{g}(\rho_1^2 - \rho_2^2 + r_0^2) \cos a \sin a.$$

One case in which this finds application is in Governors or Regulators of Steam Engines, where we have a conical pendulum, formed by attaching two heavy iron bolts to two rods, which open at a greater angle the greater the speed of the engine. This pendulum is connected by means of a train of mechanism with the valve which admits steam to the engine, and thus the valve is partially closed, diminishing the speed of the engine.

As a general rule, when over-nicety is not required, the rods are totally neglected, and the whole weight is supposed to be concentrated at the centres of the balls, so that the calculations are made as if for a simple conical pendulum.

Note on Art. 607.

Let Fig. 47 represent the pendulum, and let A be the point of suspension, B the centre of oscillation; and suppose the impulse of the ball to cause the pendulum to rise till its centre line takes the position AC; then $i$ = angle BAC, and DB $= l$ versin $i$. Hence the velocity which the pendulum has at the point B is $V = \sqrt{(2g \cdot BD)} = \sqrt{(2gl \text{ versin } i)} = 2 \sin \frac{i}{2} \sqrt{gl}$.

In the second part of this article, where the impulse produced by the powder is to be measured, it is to be observed that V is to be calculated just as in the first case, viz., $V = \sqrt{(2gl \text{ versin } i)}$, where $l$ is the distance from the point of suspension to the centre of oscillation of the pendulum with the gun attached.

## MOTIONS OF PLIABLE BODIES.

Note on Art. 609.

For definitions of *Oscillation, Amplitude, Displacement*, refer to Art. 542, page 495. From Art. 542 we have the time of an oscillation, which is $\frac{1}{n} = 2\pi\sqrt{\left(\frac{rW}{gQ}\right)}$, where $r$ is the semiamplitude, and Q the extreme force; and if oscillations of different amplitudes are to take place in the same time, $\frac{r}{Q}$ must be a constant or $Q \propto r$, whence we have equation (1.) of this Article, viz.,

$$F = -\frac{Wa^2\delta}{g}.$$

From equation (4.) we have $a^2 = -\frac{1}{m} \cdot \frac{F}{\delta}$, whence Rankine's final statement.

Note on Art. 610.

Equations (1.) are derived from the principles of the strength of materials which I shall not discuss here, but I shall merely explain what the different letters mean.

$\delta$ is the maximum deflection caused by the load F (Fig. 48).

$b$ is the breadth of a cross section of the beam.

$h$ is the height of the cross section.

E what is known as the coefficient of elasticity (a constant depending on the nature of the material, and determined by experiment).

$n'$ and $n'''$ constants depending on the distribution of the load; hence in the equation, $F = -\frac{n'Ebh^3}{n'''c^3} \cdot \delta$, the factor $\frac{n'Ebh^3}{n'''c^3}$ depends entirely on the form of the spring, the material of which it is composed, and the *manner* of *distributing* the load; and hence may be represented by $f$, as Rankine does. $\therefore F = -f\delta$.

From equation (4.), Art. 609, $a = \sqrt{\frac{g}{W}\left(-\frac{F}{\delta}\right)}$, and from Art. 610, $F = -f\delta \therefore -\frac{F}{\delta} = f \therefore a = \sqrt{\frac{fg}{W}}$.

In the case of gyration, by referring to equation (5.), Art. 598, we have for the moment of the couple,

$$M = \frac{Ik^2 i}{g}, \text{ or } \frac{Ik^2\delta}{g}, \text{ and this must } = Fl = f\delta l.$$

$$\therefore \frac{Ik^2}{g} = fl \therefore k = \sqrt{\frac{flg}{I}}.$$

With Art. 611 compare Art. 543, Rankine.

If $n_x = 2n_y$, then during the time occupied by the body in making one complete oscillation in the direction $Oy$, it makes two in the direction $Ox$; hence we must have a two lobed curve, and, from symmetry, the point of crossing of the lobes must be on the axis $Ox$.

Note on Art. 613, Ex. I.

In making use of the general equations, as given in this Article, it is important to keep in mind the distinctions between the terms used.

(*a*.) Stresses are the forces that cause the displacements.

(*b*.) Strains are the magnitudes of the displacements themselves.

(*c*.) The coefficient of elasticity of a substance is the force that would cause in a prism of the substance of a cross section of a

unit of area, an extension equal to its own length (or, in other words, extend it to double its length).

Hence if the strains be expressed as fractions of the length of the body, the stress which produces a given strain will be obtained by multiplying the strain by the coefficient of elasticity of the body. In equations (4.) and (5.) of this article, $d\xi$ expresses the actual magnitude of the strain, while $dx$ expresses the length of the molecule; hence the strain expressed as a fraction of the length of the particle is

$$\alpha = \frac{d\xi}{dx} = \frac{d\delta}{dx}\cos\theta, \text{ and so } \nu = \frac{d\eta}{dx} = \frac{d\delta}{dx}\sin\theta.$$

The stress required to produce the strain $\alpha$ will therefore be

$$p_{xx} = A\alpha = A\frac{d\xi}{dx} = A\frac{d\delta}{dx}\cos\theta;$$

and to produce the strain $\nu$,

$$p_{xy} = C\nu = C\frac{d\delta}{dx}\sin\theta.$$

In this case, equations (2.) of Art. 116, altered as shown in Ex. I. of this Article, become,

$$\frac{dp_{xx}}{dx} = \frac{w}{g}\frac{d^2\xi}{dt^2}, \text{ and } \frac{dp_{xy}}{dx} = \frac{w}{g}\frac{d^2\eta}{dt^2}; \text{ hence we have,}$$

$$Q_x = A\frac{d^2\xi}{dx^2} = A\frac{d^2\delta}{dx^2}\cos\theta = \frac{w}{g}\frac{d^2\xi}{dt^2}.$$

$$Q_y = C\frac{d^2\eta}{dx^2} = C\frac{d^2\delta}{dx^2}\sin\theta = \frac{w}{g}\frac{d^2\eta}{dt^2}.$$

which equations, as Rankine states, reduce to

$$\frac{d^2\xi}{dt^2} = a^2\frac{d^2\xi}{dx^2}, \text{ and } \frac{d^2\eta}{dt^2} = c^2\frac{d^2\eta}{dt^2}\ \therefore$$

the first of equations (12.) of Rankine, viz.,

$$\xi = \varphi(at + x) + \varphi(at - x) \text{ gives}$$

$$\frac{d\xi}{dt} = a\varphi'(at + x) + a\varphi'(at - x), \text{ and}$$

$$\frac{d\xi}{dx} = \varphi'(at + x) - \varphi'(at - x); \text{ and again,}$$

$$\frac{d^2\xi}{dt^2} = a^2\{\varphi''(at + x) + \varphi''(at - x)\}, \text{ and}$$

$$\frac{d^2\xi}{dx^2} = \varphi''(at+x) + \varphi''(\text{at}-x) \ \therefore \ \frac{d^2\xi}{dt^2} = a^2\frac{d^2\xi}{dx^2}; \text{ so also}$$

$$\frac{d^2\eta}{dt^2} = c^2\frac{d^2\eta}{dx^2} \text{ gives } \eta = x(ct+x) + w(ct-x).$$

Equations (14.) and (15.) should be

$$\frac{1}{a^2}\frac{d^2\xi}{dt^2} = \frac{d^2\xi}{dx^2} = -b^2\xi.$$

$$\frac{1}{c^2}\frac{d^2\eta}{dt^2} = \frac{d^2\eta}{dx^2} = -b'^2\eta.$$

To deduce them we have the moving force $= \frac{d^2\xi}{dt^2} = a^2\frac{d^2\xi}{dx^2}$, and the displacement $\xi$, $\therefore$ since the moving force is proportional to the displacement

$$\frac{\frac{d^2\xi}{dt^2}}{-\xi} = a^2b^2 = a \text{ constant}, \therefore \frac{1}{a^2}\frac{d^2\xi}{dt^2} = -b^2\xi. \quad \text{Q. E. D.},$$

$$\text{so also } \frac{1}{a^2}\frac{d^2\eta}{dt^2} = \frac{d^2\eta}{dx^2} = -b'^2\eta.$$

The following is the mode of deducing equations (16.) and (17.).

From (14.) we have,

$$\frac{d^2\xi}{dx^2} = -b^2\xi \ \therefore \ 2\frac{d\xi}{dx}\frac{d^2\xi}{d^2x}dx = -2b^2\xi\frac{d\xi}{dx}dx \ \therefore \left(\frac{d\xi}{dx}\right)^2 = -b^2\xi^2 + \text{C},$$

which may be expressed thus:

$$\left(\frac{d\xi}{dx}\right)^2 = -b^2\xi^2 + b^2c^2\xi_1^{\,2} = b^2(c^2\xi_1^{\,2} - \xi^2)$$

$$\therefore \frac{d\xi}{\sqrt{(c^2\xi_1^{\,2} - \xi^2)}} = \pm b dx \ \therefore \ \cos^{-1}\frac{\xi}{c\xi_1} = bx - c_1$$

$$\therefore \frac{\xi}{c\xi_1} = \cos(bx - c_1). \quad (1.)$$

Again from $\frac{d^2\xi}{dt^2} = -a^2b^2\xi$, we should derive in a similar manner

$$\frac{\xi}{k\xi_1} = \cos(abt - k_1). \quad (2.)$$

From (1.), $\quad \frac{\xi}{\xi_1} = c\cos(bx - c_1)$,

and from (2.) $\quad \frac{\xi}{\xi_1} = k\cos(abt - k_1)$,

where $c$ may be a function of $t$, but not of $x$, and $k$ may be a function of $x$, but not of $t$. Both of these are satisfied if we have,

$$\frac{\xi}{\xi_1} = \cos(bx - c_1)\cos(abt - k_1).$$

The constants $c_1$ and $k_1$, are determined from the conditions that when $\xi = \xi_1$, $x = x_0$, and $t = t_0$. When $\xi = \xi_1$ we have

$$1 = \cos(bx_0 - c_1)\cos(abt_0 - k_1);$$

hence we have $\cos(bx_0 - c_1) = 1$, and $\cos(abt_0 - k_1) = 1$, since neither can be greater than 1;

$$\therefore bx_0 - c_1 = 0, \text{ and } abt_0 - k_1 = 0,$$

$$\therefore c_1 = bx_0, \text{ and } k_1 = abt_0,$$

$$\therefore \frac{\xi}{\xi_1} = \cos b(x - x_0)\cos ab(t - t^0),$$

$$\text{or } \xi = \xi_1 \cos\frac{2\pi}{\lambda}(x - x_0)\cos\frac{2\pi a}{\lambda}(t - t_0);$$

so also, $$\eta = \eta_1 \cos\frac{2\pi}{\lambda}(x - x'_0)\cos\frac{2\pi c}{\lambda}(t - t'^0).$$

Nodal Planes. Make $\xi = 0$, in equation (16.), Rankine, and we have,

$$\cos\frac{2\pi}{\lambda}(x - x_0) = 0 \therefore \frac{2\pi}{\lambda}(x - x_0) = (2n + 1)\frac{\pi}{2}$$

$$\therefore x - x^0 = (2n + 1)\frac{\lambda}{4}, \text{ an odd number of times } \frac{\lambda}{4}.$$

Ventral Planes. Make $\xi = \xi_1$ in equation (16.), Rankine, and we have

$$\cos\frac{2\pi}{\lambda}(x - x_0) = 1 \therefore \frac{2\pi}{\lambda}(x - x_0) = n\pi \therefore x - x_0 = n\frac{\lambda}{2}.$$

Before learning Art. 614, the student should first learn Art. 416, page 416, of Rankine.

Note on Art. 615.

The demonstration of Equation (1.) would require principles which have not yet been explained; it may be found in Rankine's treatise on the Steam Engine, and other Prime Motors, page 319, Art. 251.

To prove Equation (4.), refer to Art. 122 of the Applied Mechanics, and the corresponding explanation on page 96 of the first part of this treatise.

Equation (6.) is obtained by substitution from (2.) of 614, and (2.) and (4.) of 615.

Note on Art. 616.

$\frac{R}{S}$ is the greatest force that acts on a unit of area of the cross section; hence if the compression produced by this force be represented by $c$, we have,

$$c : L = \frac{R}{S} : E \therefore c = \frac{RL}{ES}.$$

EXAMPLE.

1. Given a pile made of red oak, where S = 3 square feet; of length = 50 ft. when W = 500 lbs., what is the effective resistance of the ground when a fall of the ram of 40 ft. drives the pile 1 ft. into the ground.

## MOTION OF FLUIDS.

In order to understand Articles 618–619 and 620, the student should first read Articles 404, 413, 414 and 415, of Rankine.

Note on Art. 619.

To illustrate what is meant by "Dynamic Head," suppose in the case of still water, that O, A and P (Fig. 49), represent respectively the position of the datum plane, the free upper level and the particle of liquid under consideration. Now if the pressure per unit of area (disregarding the pressure of the air) be $p$, the height due to this pressure is $\frac{p}{\rho}$, and will, in this case, be AP.

On the other hand, $z = OP$ is the height of the particle above the datum plane, and this is the height due to the increase of pressure that the particle would have were it to descend to O; the sum of the two, or AO, is the Dynamic Head, and is the head corresponding to the amount of work that the particle could do at O, or, in other words, corresponding to its potential energy; hence $h = \frac{p}{\rho} + z$ is the dynamic head.

In the case of still water this is the whole head, but if the water is moving with a velocity $v$, it would be able to perform an additional amount of work.

The case when $h = \frac{p}{\rho} - z$ is when the datum line is situated as in Fig. 50.

In connection with the above, and with Arts. 621 and 622, I would refer the student to Rankine's Steam Engine and other Prime Motors, Art. 98.

Note on Art. 620.

From Equation (1.), Art. 619, we have,

$$p - \rho z = \rho h \therefore \frac{dp}{dx} = \rho \frac{dh}{dx} \therefore -\frac{dp}{\rho dx} = -\frac{dh}{dx};$$

hence follows Equations (1.) of this Article.

Note on Art. 621.

By referring to Arts. 413 and 415 of Rankine, we find,

$$u = \frac{d\xi}{dt} = \frac{dx}{dt}; \ v = \frac{d\eta}{dt} = \frac{dy}{dt}; \text{ and } w = \frac{d\zeta}{dt} = \frac{dz}{dt};$$

$$\therefore d\frac{V^2}{2g} = -\left(\frac{dh}{dx}\frac{d\xi}{dt} + \frac{dh}{dy}\frac{d\eta}{dt} + \frac{dh}{dz}\frac{d\zeta}{dt}\right)dt$$

$$= -\left(\frac{dh}{dx}\frac{dx}{dt} + \frac{dh}{dy}\frac{d\eta}{dt} + \frac{dh}{dz}\frac{dz}{dt}\right)dt = -\frac{dh}{dt}dt = -dh,$$

$$\therefore \frac{V^2}{2g} + h = \text{a constant.}$$

Where $\frac{V^2}{2g}$ is the height due to the velocity $v$, and $h$ is the dynamic head.

Note on Art. 622.

If in Fig. 49 the particle at P is moving with a velocity V, it has, in addition to the potential energy above mentioned, which corresponds to its dynamic head, an actual energy $\frac{WV^2}{2g}$, and the total amount of work it is capable of doing is,

$$W\left(\frac{p}{\rho} + z + \frac{V^2}{2g}\right) = Wh + W\frac{V^2}{2g},$$

and its total head is $h + \frac{V^2}{2g}$.

This is very fully explained in Art. 98 of Rankine's Steam Engine.

The following example, taken from Weisbach, will serve to illustrate Art. 625.

If water flows from a vessel whose cross section is 60 square inches through a circular orifice in the bottom, 5 inches in diameter under a head of water of 6 feet, what is its velocity of efflux?

Note on Art. 627.

To deduce Equation (2.) imagine the orifice divided into a number of small horizontal bands, the area of each one being $\Delta A$, its width $y$, and its depth $\Delta z$, then will the amount of water discharged (disregarding all disturbing influences), be,

$$\Delta Q = \Delta A \sqrt{\{2g(z_1 - z)\}} = \sqrt{(2g)}\, \Delta A \sqrt{(z_1 - z)}.$$

If $z_1 - z_0$ denotes the mean head, we must have,

$$Q = \sqrt{(2g)}\, \Sigma \Delta A \sqrt{(z_1 - z)} = \sqrt{(2g)} \sqrt{(z_1 - z_0)}\, \Sigma \Delta A.$$

$$\therefore \sqrt{(z_1 - z_0)} = \frac{\Sigma \sqrt{(z_1 - z)}\, \Delta A}{\Sigma \Delta A} = \frac{\int_{z'}^{z''} \sqrt{(z_1 - z)}\, y dz}{\int_{z'}^{z''} y dz}$$

$$\therefore z_1 - z_0 = \left\{ \frac{\int_{z'}^{z''} y \sqrt{(z_1 - z)}\, dz}{\int_{z'}^{z''} y dz} \right\}^2$$

Equation (3.) is derived directly from (2.) by integration, considering $y$ as constant.

EXAMPLE.

What is the mean head for a rectangular orifice of efflux 3 ft. wide and $1\frac{1}{4}$ feet high, the lower edge being $2\frac{3}{4}$ feet below the level of the water.

Note on Art. 628.

To fix the ideas, imagine a set of concentric cylinders to be the surfaces of equal head, *i. e.*, that the dynamic head is the same throughout each of these cylinders, while the head is different in one cylinder from that in another. This is illustrated in Rankine,

Fig. 250. Now the force due to the difference of head in the different cylinders cannot have any component in the direction of the cylinder itself: hence if the deviating force is to be furnished entirely by the difference of head, the motion must take place in a plane perpendicular to the surface of the cylinder, or as Rankine expresses it, the radius of curvature must be in a plane perpendicular to the surface of equal head.

To deduce Equations (2.) we have as follows:

1. The deviating force for mass $m$ is $\frac{mV^2}{gr}$.

2. The variation of head per unit of distance normal to the surface of equal head $= -\frac{dh}{dn}$, and this into the mass, is proportional to the force acting in the direction of the above said normal.

The component of this in the direction of the radius of curvature is $-m\frac{dh}{dn}\cos nr$ ∴

$$m\frac{V^2}{gr} = -\frac{dh}{dn}\cos nr \therefore \frac{V^2}{gr} = -\frac{dh}{dn}\cos nr. \quad \text{Q. E D.}$$

Note on Art. 29.

To deduce Equation (1.), referring back to Art. 407, we have, $v =$ the mean radial component of the velocity. Now $2\pi rb$ is the portion of the surface of the cylinder through which the current passes, and $v2\pi rb$ will consequently represent the volume discharged per hour from the surface of the cylinder,

$$\therefore v\cdot 2\pi rb = Q \therefore v = \frac{Q}{2\pi rb}.$$

Equation (2.) is deduced from (4.), Art. 625, and (1.) of this article.

Note on Art. 630.

To deduce Equatiou (3.); from $\frac{dh}{dr} = \frac{2(h_1 - h)}{r}$, we have, first,

$$\frac{dh}{h_1 - h} = \frac{2dr}{r} \therefore -\log(h_1 - h) = 2\log r - \log c,$$

$$\therefore \log\frac{1}{h_1 - h} = \log\frac{r^2}{c}, \text{ or } h_1 - h = \frac{c}{r^2} \therefore h_1 - h \propto \frac{1}{r^2},$$

$$\text{or } \frac{v^2}{2g} \propto \frac{1}{r^2} \therefore v \propto \frac{1}{r}. \quad \text{Q. E. D.}$$

Note on Art. 633. Equation (1.) of Rankine should be

$$h_1 = 2h_0 = \frac{v_0^2}{g}.$$

The variables in Equation (3.) are $z$ and $r$, hence the equation might be written $z = cr^2$, or $r^2 = \frac{z}{c}$, which is the equation of a parabola, whose vertex is at O, and tangent to OA. When the water arrives at B it will have acquired a velocity due to the height BC. In moving in the forced vertex, this velocity will be gradually diminished in direct proportion to its distance from O$z$; hence the velocity lost when it arrives at any point of its path, will be that due to the depth of the point below B; or, in other words, the velocity which it will still have will be that due to its height above AA, since at O its velocity will be 0.

In connection with Art. 639 the student may read Art. 161 of Rankine's treatise on the Steam Engine and other Prime Movers.

## MOTIONS OF GASES.

Note on Art. 635.

To deduce Equation (3.) from (1.) we have,

$$p = \rho^\gamma \therefore dp = \gamma\rho^{\gamma-1} d\rho \therefore \frac{dp}{\rho} = \gamma\rho^{\gamma-2} d\rho$$

$$\therefore h = \int_0^p \frac{dp}{\rho} + z = \gamma \int_0^p \rho^{\gamma-2} d\rho + z$$

$$\therefore h - z = \gamma\frac{\rho^{\gamma-1}}{\gamma - 1} = \frac{\gamma}{\gamma-1}\frac{\rho^\gamma}{\rho} = \frac{\gamma}{\gamma-1}\frac{p}{\rho}.$$

In connection with Equation (7.) I would refer the student to Equation (5.), page 564, of Rankine.

Note on Art. 637.

From Equations (11.), Art. 635, we have,

$$\tau \propto \rho^{\gamma-1} \propto p^{\frac{\gamma-1}{\gamma}} \therefore c\tau = p^{\frac{\gamma-1}{\gamma}} \therefore c\tau_2 = p_2^{\frac{\gamma-1}{\gamma}}, \text{ and}$$

$$c\tau_1 = p_1^{\frac{\gamma-1}{\gamma}}.$$

$$\therefore \frac{\tau_2}{\tau_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma - 1}{\gamma}}; \text{ so also } \frac{\tau_2}{\tau_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma - 1}$$

$$\left(\frac{\rho_2}{\rho_1}\right)^{\gamma - 1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma - 1}{\gamma}} \therefore \frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}}.$$

All the remaining equations are easily derived.

Note on Art. 637, A.

To determine when the flow of weight is a maximum, let $\frac{p_2}{p_1} = x$, then since from (4.) of 637,

$$\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}} \sqrt{\left(1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma - 1}{\gamma}}\right)} = \text{max., we have}$$

$$x^{\frac{1}{\gamma}} \sqrt{\left(1 - x^{\frac{\gamma - 1}{\gamma}}\right)} = \text{max., or } x^{\frac{2}{\gamma}}\left(1 - x^{\frac{\gamma - 1}{\gamma}}\right) = x^{\frac{2}{\gamma}} - x^{\frac{\gamma + 1}{\gamma}} = \text{max.;}$$

now differentiate, and we have

$$\frac{2}{\gamma} x^{\frac{2}{\gamma} - 1} - \left(1 + \frac{1}{\gamma}\right) x^{\frac{1}{\gamma}} = 0 \therefore \frac{2}{\gamma} x^{\frac{1}{\gamma} - 1} = \frac{1 + \gamma}{\gamma}$$

$$\therefore x^{\frac{1 - \gamma}{\gamma}} = \frac{\gamma + 1}{2} \therefore x^{\frac{\gamma - 1}{\gamma}} = \frac{2}{\gamma + 1} \therefore x = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}}$$

hence follow the remaining equations deduced.

To deduce Equation (8.) we have,

$$v = \text{velocity of sound} \times \sqrt{\left(\frac{2}{\gamma + 1}\right)}, \text{ and } \frac{\rho_2}{\rho_1} = \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma - 1}}$$

$$\therefore \frac{v\rho_2}{\rho_1} = \text{velocity of sound} \times \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{2} + \frac{1}{\gamma - 1}} =$$

$$\text{velocity of sound} \times \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma + 1}{2(\gamma - 1)}}.$$

## MOTION OF LIQUIDS WITH FRICTION.

### EXAMPLES.

1. Suppose the cross section of the channel of an open stream to be a semicircle (radius 20 ft.), what will be the angle of declivity for a velocity of 10 feet per second? (Use the value $f = a + \frac{b}{v}$, as given in the text.)

2. What will be the declivity when the cross section of the above stream is a trapezoid; breadth at top water = 24 ft.; breadth at bottom = 12 ft. Depth of water = 4 ft.

3. What is the hydraulic declivity in a uniform pipe of $1\frac{1}{2}$ inch diameter, where $v = 8$ ft per second?

4. What in a 2 inch pipe?

## IMPULSE OF FLUIDS AND SOLIDS.

Note on Art. 648.

The jet is at first moving with the velocity $v$ in the direction $ab$ (Fig. 50), and after leaving the surface it is moving in a direction $ow$, parallel to $ac$, with the same velocity $v$. To find the deviating force we follow the method of Art. 363 of Rankine, and laying off $ab$ and $ac$, both equal to $v$, the line $bc$ will represent the change of velocity in magnitude and direction, and

$$bc = 2be = 2ab \sin \frac{\beta}{2} = 2v \sin \frac{\beta}{2}.$$

The components of this velocity along and perpendicular to the original direction of the jet, will be represented by $bd$ and $dc$, respectively, and from the figure we have

$$bd = ab - ad = v - v \cos \beta, \text{ or } v_x = v\,(1 - \cos \beta), \text{ and}$$

$$dc = v \sin \beta, \text{ or } v_y = v \sin \beta \;\therefore$$

$$F : F_x : F_y = 2 \sin \frac{\beta}{2} : 1 - \cos \beta : \sin \beta$$

$$\therefore F_x = \frac{F(1 - \cos \beta)}{2 \sin \frac{\beta}{2}}, \quad F_y = \frac{F \sin \beta}{2 \sin \frac{\beta}{2}}, \text{ or}$$

$$F_x = \frac{\rho Q v}{g}(1 - \cos \beta); \; F_y = \frac{\rho Q v}{g} \sin \beta. \quad \text{Q. E. D.}$$

The above applies to a jet impinging on a smooth surface which is itself at rest. The case which applies to water wheels, where the surface on which the water impinges is in motion, is treated of in Art. 649 of Rankine.

Note on Art. 649.

When the surface (as the vane of a water wheel) has a motion of translation in the direction BD, as shown in Fig. 254 of Rankine, the direction of motion and the velocity of the water relatively to surface will be represented by DC, the resultant of the motion of the water and of the surface: hence the isosceles triangle, by which we determine the velocity of deflection will have one side parallel to DC, and the other to the tangent EF, and the two equal sides will each be equal to DC.

To prove Equation (10.) we have

$$\left.\begin{array}{l} v_1^2 = w^2 - u^2 + 2wv_1 \cos a \\ v_2^2 = w^2 + u^2 + 2wu \cos \gamma \end{array}\right\} \text{ hence by subtraction,}$$

$$v_1^2 - v_2^2 = 2u(v_1 \cos a - u - w \cos \gamma)$$

$$\therefore \ Fxu = \frac{\rho Q}{2g} 2u(v_1 \cos a - u - w \cos \gamma),$$

which is the same value as that obtained in Equation (8.).

Note on Art. 650.

Q being the quantity of water discharged, $b$ the depth of the wheel, and $r_1$ the radius of the cylindrical surface of discharge; the area of this surface is $2\pi r_1 b$, and hence the radial velocity of the issuing current is $u = \dfrac{Q}{2\pi r_1 b}$.

Now as the cylindrical surface of discharge is moving with a velocity $ar_1$, and the velocity of the issuing jet at right angles to its motion (in a radial direction) is $u$, in order that the guide plates may be inclined in the direction of the resultant motion of the water we must have $u \tan \theta = ar_1$, or $\tan \theta = \dfrac{ar_1}{u}$.

Note on Art. 656.

The changes referred to in this Article are those that occur in the steam contained in the cylinder of a steam engine, and the curves represented in the figures are such as would be obtained on the Indicator card: their areas represent the energy transferred to the piston from the steam, while the abscissæ represent the respective volumes occupied by the steam at different periods in the

course of the piston, and the ordinates the corresponding intensities of pressure.

To prove Equation (4.) we have from Art. 635, Equation 2,

$$p \propto \rho^{\gamma}, \text{ or } p = c\rho^{\gamma} \therefore dp = c\gamma\rho^{\gamma-1}d\rho \therefore$$

$$\int_{p_2}^{p_1}\frac{dp}{\rho} = \gamma c\int_{\rho_2}^{\rho_1}\rho^{\gamma-2}\,d\rho = \frac{\gamma}{\gamma-1}\left\{\rho^{\gamma-1}\right\}_{\rho_2}^{\rho_1} = \frac{\gamma c}{\gamma-1}\left(\rho_1^{\gamma-1}-\rho_2^{\gamma-1}\right)$$

or $$\int_{p_2}^{p_1}\frac{dp}{\rho} = \frac{\gamma}{\gamma-1}\left(\frac{p_1}{\rho_1}-\frac{p_2}{\rho_2}\right) = \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1}\left(1-\frac{p_2\rho_1}{p_1\rho_2}\right);$$

but $$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} \therefore \int_{p_2}^{p_1}\frac{dp}{\rho} = \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1}\left\{1-\left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1}\right\}$$

$$\therefore \int_{p_2}^{p_1} s\,dp = \frac{\gamma}{\gamma-1}s_1p_1\left\{1-\left(\frac{s_1}{s_2}\right)^{\gamma-1}\right\}. \quad \text{Q. E. D.}$$

To deduce Equations (7.) from Equations (6.), we have

$$\frac{dp}{\rho} = \left(\frac{B}{z^2}+\frac{2C}{z^3}\right)dz \therefore$$

$$z\frac{dp}{dz} = zp\left(\frac{B}{z^2}+\frac{2C}{z^3}\right) = p\left(\frac{B}{z}+\frac{2C}{z^2}\right).$$

# NOTES ON MECHANICS.

## PART II.—DYNAMICS.

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